Synthetic CDO pricing using the double normal inverse Gaussian copula with stochastic factor loadings

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Summary

Collateralized Debt Obligations (CDOs) are credit derivatives that have gained interest in recent years, both from the market side, because of a dramatic increase in traded contracts, as well as from an academic side because the pricing of such contracts is difficult and still an open issue.

At a very simple level a collateralized debt obligation, is a transaction that transfers the credit risk of a reference portfolio of assets. The defining feature of a CDO structure is the tranching of credit risk. The risk of loss on the reference portfolio is divided into tranches of increasing seniority. Losses will first affect the 'equity' or 'first loss' tranche, next the 'mezzanine' tranches, and finally the 'senior' tranches. In this thesis the pricing of tranches of synthetic CDOs is studied. In a synthetic CDO the reference portfolio consists of credit default swaps.

Chapter 1 explains some basic aspects of CDOs, such as trading strategies, leverage, and CDO indices. The general approach to pricing a CDO tranche is introduced in Chapter 2. It shows that the CDO pricing problem is solved as soon as the loss distribution of the reference portfolio can be calculated. In Chapter 3 the Gauss copula model for loss distribution modelling is introduced. The Gauss copula model is the approach most often applied by practitioners. The large portfolio approximation is introduced as well. In Chapter 4 some issues arising by applying the Gauss copula model for CDO tranche pricing are discussed. This Chapter shows why a trader relying on the Gauss copula model should be very careful. Some extensions to the Gaussian copula model are reviewed in Chapter 5. In Chapter 6, CDO pricing using two extensions to the Gauss copula model, the double normal inverse Gaussian model (double NIG model) and the Gauss model with stochastic factor loadings, are explained in detail. Additionally, a new extension to the Gauss copula model is developed: the double normal inverse Gaussian model with stochastic factor loadings. In Chapter 7 the numerical results of pricing tranches of the DJ iTraxx with the four models introduced in Chapter 6 are compared.

In summary, all the three tested extensions to the Gauss one factor model significantly improved the fit to market data. Even though the double normal inverse Gaussian model with stochastic factor loadings produced the best fit, for CDO pricing the simple double NIG model or the Gauss stochastic factor loadings model may be preferred by practitioners due to the greater numerical efficiency.

Chapter 1

CDO Basics

1.1 What is a Collateralized Debt Obligation?

At a very simple level a collateralized debt obligation or CDO, is a transaction that transfers the credit risk of a reference portfolio of assets. The defining feature of a CDO structure is the tranching of credit risk. The risk of loss on the reference portfolio is divided into tranches of increasing seniority. Losses will first affect the 'equity' or 'first loss' tranche, next the 'mezzanine' tranches, and finally the 'senior' tranches. For example the equity tranche bears the first 3% of the losses, the second tranche bears 3% to 6% of the losses and so on. When tranches are issued, they usually receive a rating from an independent agency (Moody's, S&P, Fitch etc.).

By tranching the loss different classes of securities are created, which have varying degrees of seniority and risk exposure and are therefore able to meet very specific risk return profiles of investors. Investors take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn collecting the premium.

It is common to distinguish between synthetic and cash CDOs. Cash CDOs have a reference portfolio made up of cash assets, such as corporate bonds or loans. In a synthetic CDO the reference portfolio consists of credit default swaps. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon.

1.2 Why are CDOs issued?

The possibility to buy CDO tranches is very interesting for investors to manage credit risk. The investment in a CDO tranche with a specific risk-return profile is much more attractive for a credit investor or a hedger than to achieve the same goal via the rather illiquid bond and loan market with high bid/ask spreads.

However, it may not always be immediately clear why CDOs are issued at all, since the costs of lawyers, issuers, asset managers and rating agencies encountered when setting up a CDO can be very high. (For the role that lawyers play in the CDO business see Wolcott [27]). Besides the reason mentioned above, that by tranching one creates securities fitting very specific risk appetites of investors, there are two main reasons why CDOs are issued, which are discussed in the following.

1.2.1 Spread arbitrage opportunity

Imagine that the portfolio of a hypothetic CDO consists entirely of credit default swaps (CDS in the following). The CDO issuer bought the single name CDS and will receive on each name a premium. With these premia the CDO issuer pays itself premia to the CDO tranche holders. The goal of the spread arbitrage is that the total spread collected from the single name CDS exceeds the total spread to be paid to investors of the CDO tranches. Such a mismatch typically creates a significant arbitrage potential which offers an attractive excess spread to equity and subordinated notes investors.

1.2.2 Regulatory capital relief or balance sheet CDOs

Balance sheet CDOs are initiated by holders of securitizable assets, such as commercial banks, which desire to sell assets or transfer the risk of assets. The motivation may be to shrink the balance sheet, reduce regulatory capital, or reduce required economic capital.

In simple terms such a transaction works in the following way: In general, loan pools require regulatory capital in size of 8% times Risk Weighted Assets of the reference pool (according to Basel II standard model). After the securitization of the pool, the only regulatory capital requirement the originator has to provide regarding the securitized loan pool is holding capital for retained pieces. For example if the originator retained the equity tranche, the regulatory capital required on the pool would have been reduced from 8% to 50bp, which is the size of the equity tranche. The 50bp come from the fact that retained equity pieces usually require full capital deduction.

As nice as this looks at first sight one has to keep in mind that the costs for capital relief are high. The originating bank has to pay premia to note holders, upfront costs for lawyers, rating agencies and structuring, and ongoing administration costs. A thorough calculation of the costs of issuing the CDO compared to the relief of reducing the regulatory capital (and thereby maybe avoid a downgrading of the firm rating) is required to decide on such a major transaction.

Remark 1.2.1 *Risk transfer in life insurance. As mentioned above bank securitization transactions can reduce regulatory capital. A regulatory capital relief is definitely also of interest to the life insurance industry. In this remark it is discussed if and how the CDO framework could be adapted to fit a life insurance securitization transaction.*

The risks that are ideal to securitize and sell as tranched notes are the ones which are easy to quantify. Over the years life insurances have become experts in the calculation of mortality and longevity risk. We can therefore speak of those risks as easy to quantify. One of the main differences between life insurance and commercial banks is that the liabilities, i.e. insurance contracts are not a tradable asset, which leaves a life insurance not much room for optimizing its liability portfolio via the market. Reinsurance has accessed the market recently via so called cat-bonds (catastrophy bonds) and successfully transferred extreme risks to the financial market.

One way in which a life insurance could sell typically highly illiquid insurance risk to the market is via tranching of a portfolio of liabilities, which is very much in the spirit of a CDO for commercial banks. Even though such a risk transfer is obvious when thinking in the CDO framework, tranching has not yet been fully exploited in the insurance securitization market. A review of the existing examples can be found in [8] and [24].

The reason for the reluctance of the life insurance industry may stem from different sources. For such a securitization transaction to be profitable, it has to be large. Otherwise the costs of setting up the transaction will exceed the benefits. Most life insurances may simply not have the size to set up a profitable transaction. Another serious problem is regulation. Adverse regulatory decisions could create serious problems for securitization. Moreover, it is unclear if investors are ready to invest in insurance risks, as they are not familiar with this risk class. On the other hand a new risk class could be very interesting to the sophisticated investor as it provides a new diversification possibility, which is much more transparent and specific in the risk-return profile than the alternative, the direct investment in the life insurance stock.

The possibility of a reduction of regulatory capital through securitization via *tranching should definitely be considered and studied thoroughly. Especially, if the required regulatory capital is high compared to the available, a securitization transaction increasing available equity capital can be seen as an alternative to a costly capital increase via issuance of new stock and may prevent a downgrading of the rating.*

1.3 Synthetic CDOs and credit-indices

It is obvious that setting up a arbitrage or balance sheet CDO is very complicated. To price such custom made CDOs a highly complex model has to be built which exactly matches the peculiarities of each CDO in question. For example, we would have to analyse the trading behaviour of the CDO portfolio manager, which would include a microeconomic analysis of utility functions and probably also concepts from game theory. Therefore, in order to not loose track of the question that lies at the heart of the CDO pricing problem, we are here interested in pricing *synthetic* CDOs. In a synthetic CDO the reference portfolio consists of credit default swaps and is static (not managed). These CDOs became of great interest in recent years because the major market makers produced common synthetic CDO indices which dramatically increased price transparency and market liquidity.

	tranched CDS indices	CDS indices
June 2004	20 bn	100 bn
June 2005	over 100 bn	over 450 bn

Table 1.1: Outstanding notional on tranched CDS indices and on CDS indices. (Indices: DJ iTraxx, DJ CDX).

The credit derivatives business has grown dramatically in recent years. The introduction of common synthetic indices considerably added to this growth. Table 1.1 shows the change in outstanding notional on synthetic CDO tranches and CDS indices. CDS are the main plain-vanilla product in the market. Reported numbers of outstanding notional in the whole credit derivatives market reached USD 2 trillion in 2003 [20].

1.3.1 DJ iTraxx Europe

In June 2004, the DJ iTraxx Europe index 'family' was created by merging existing credit indices (www.iboxx.com), thereby providing a common platform to all credit investors. The DJ iTraxx Europe consists of a static portfolio of the top 125 names in terms of CDS volume traded in the six months prior to the roll. Each name is equally weighted in the static portfolio. A new series of DJ iTraxx Europe is issued every 6 months. This standardisation led to a major increase in transparency and liquidity of the credit derivatives market(see Table 1.1). The new index allows for a cost efficient and timely access to diversified European credit market and is therefore attractive for portfolio managers, as a hedging tool for insurances and corporate treasuries as well as for credit correlation trading desks.

Besides a direct investment in the DJ iTraxx Europe index via a CDS on the index or on a subindex, it is also possible to invest in standardized tranches of the DJ iTraxx Europe index via the DJ tranched iTraxx which is nothing else but a synthetic CDO on a static portfolio.

Reference Portfolio	Tranche name	K_A	K_D	Tranche number
DJ iTraxx Europe: Portfolio of 125 CDS	Equity	0 %	3 %	1
	Junior Mezzanine	3 %	6 %	2
	Senior Mezzanine	6 %	9 %	3
	Senior	9 %	12 %	4
	Super Senior	12 %	22 %	5

Table 1.2: Standard synthetic CDO structure on DJ iTraxx Europe. With K_A and K_D, the loss attachment and detachment points.

1.4 Leverage

To get familiar with the riskiness of a CDO tranche credit derivative investment an illustrative example is outlined in this section. The leverage incurred by investing in an equity tranche of a synthetic CDO on DJ iTraxx is compared to investing directly in the DJ iTraxx index via CDS.

The CDO equity tranche investor. Consider an investor selling protection for Euro 10 mio on the DJ iTraxx equity tranche maturing on 9/2010. Assume that the DJ tranched iTraxx 0-3% trades at 600bp. The periodic premium received by the investor is therefore Euro 600'000. Without credit events the investor continues to receive the premium on the original notional of Euro 10 mio until maturity.

Now assume that one reference entity has defaulted. For simplicity zero recovery is assumed. Since the reference portfolio is the DJ iTraxx Europe which contains 125 names, the loss translates into a 0.8% loss on the 125-name portfolio. Recall that the equity tranche investor would lose the entire notional for a loss exceeding 3%. A loss of 0.8% therefore corresponds to a 27% (0.8%/3%) loss of notional. The investor pays Euro 2.7 mio to the protection buyer! The new notional is then Euro 7.3 mio and the investor will receive the 600bp premia on the new reduced notional (until any further credit event).

$$\xrightarrow{\quad CreditRisk\ \ (1st\ \ 3\,\%\ \ of\ \ losses)\quad}$$

Protection buyer	$\xrightarrow{Euro\ 600'000\ (Premium)}$	Protection seller
Protection buyer	$\xleftarrow{\textbf{Euro 2.7 mio}}$ $\xrightarrow[Euro\ 438'000\ (Premium)]{}$	Protection seller
Protection buyer	$\xleftarrow{\textbf{Euro 10 mio}}$	Protection seller

Table 1.3: CDO equity tranche investor, 10 mio notional. Cash flows before (upper row) and after (middle row) the default of one reference entity, and after the default of 4 reference entities (lower row).

In a second state of nature assume that four out of the 125 reference entities default at the same time. This is a loss of 3.2% on the 125 names portfolio. Therefore the holder of the equity tranche would lose the entire notional of Euro 10 mio to the protection buyer. The loss portion exceeding the 3% (0.2%) will now affect the holders of the next tranche absorbing the 3-6% loss.

The CDS investor. Consider an investor selling protection for Euro 10 mio on the DJ iTraxx Europe index in CDS with maturity 9/2010. This investment is equal to holding a CDO tranche on the whole portfolio (where the word tranche makes of course not much sense anymore). The CDS premium is 50bp per annum.

Now as before 1 reference entity defaults, which translates in a 0.8% loss on the 125-name portfolio. In case of the CDS this directly translates in a reduction of the notional of Euro 80'000 (0.8% x 10 mio). The new notional on which the 50bp premia will be received is then Euro 9.92 mio.

Protection buyer	$\xrightarrow{\mathit{Euro}\ 50'000\ (\mathit{Premium})}$	Protection seller
Protection buyer	$\xleftarrow{\textbf{Euro 80}'\textbf{000}}$ $\xrightarrow[\mathit{Euro}\ 49'600\ (\mathit{Premium})]{}$	Protection seller
Protection buyer	$\xleftarrow{\textbf{Euro 320}'\textbf{000}}$ $\xrightarrow[\mathit{Euro}\ 48'400\ (\mathit{Premium})]{}$	Protection seller

Table 1.4: CDS investor, 10 mio notional. Cash flows before (upper row) and after (middle row) the default of one reference entity, and after the default of 4 reference entities (lower row).

In the case when four reference entities default, the loss on the 125-name portfolio is 3.2%. The reduction of the notional is Euro 320'000. Hence the premium of 50bp will be paid on the new notional of Euro 9.68 mio.

Conclusion. By calculating this simple example of an equity tranche compared to a CDS on the index it became immediately clear that an investment in the equity tranche is highly leveraged. The loss after one credit event was 34 times higher for the equity tranche holders than for the holders of the index CDS!

This example was included to give a first gut feeling on the riskiness of CDOs. The high leverage involved is one reason why the correct pricing of CDOs is extremely important. Wrong risk assessment and identification can easily result in huge losses.

Chapter 2

General approach for pricing synthetic CDOs

Consider a CDO on a static reference portfolio consisting of credit default swaps, i.e. a synthetic CDO. As long as no credit event has happened the CDO issuer pays a regular premium to the tranche investor. In the case of a default the investor (protection seller) pays the CDO issuer (protection buyer) an amount equal to the incurred loss. The next premium is then paid on the new notional reduced by the loss amount.

2.1 Loss distribution

Recall that the default payment leg of a CDO tranche will always absorb losses that occur in the reference portfolio between two prespecified thresholds. For example the equity tranche in the example in Section 1.4 absorbs the first 3% of the losses in the reference portfolio. These thresholds are called K_A and K_D, attachment and detachment points of the CDO tranche (for illustration see Table 1.2).

Denote therefore by $M(t)$ the cumulative loss on a given tranche and by $L(t)$ the cumulative loss on the reference portfolio at time t:

$$M(t) = \begin{cases} 0 & \text{if } L(t) \leq K_A; \\ L(t) - K_A & \text{if } K_A \leq L(t) \leq K_D; \\ K_D - K_A & \text{if } L(t) \geq K_D. \end{cases}$$

The determination of the incurred portfolio loss $L(t)$ is therefore the essential part in order to calculate the cash flows between protection seller and buyer and hence in pricing the CDO tranches.

Definition 2.1 ***Portfolio loss***. *Consider N reference obligors with a nominal amount A_n and recovery rate R_n for $n = 1, 2, ..., N$. Let $L_n = (1-R_n)A_n$ denote the loss given default of obligor n. Let τ_n be the default time of obligor n. Let $N_n(t) = 1_{\{\tau_n<t\}}$ be the counting process which jumps from 0 to 1 at the default time of obligor n. Then the portfolio loss is given by:*

$$L(t) = \sum_{n=1}^{N} L_n N_n(t)$$

Note that $L(t)$ and therefore also $M(t)$ are pure jump processes. At every jump of $M(t)$ a default payment has to be made from the protection seller to the protection buyer.

In the following we will assume that the notional amount A_n and the recovery rate R_n are the same for all obligors. In discrete time we can then write the expected percentage cumulative loss on a given tranche as:

$$\begin{aligned} EL_{(K_A,K_D)}(t_i) &= \frac{\mathbb{E}\left[M(t_i)\right]}{K_D - K_A} \\ &= \frac{1}{K_D - K_A} \sum_{n=1}^{N} \left(min(L_n(t_i), K_D) - K_A\right)^+ \, p_n \end{aligned}$$

Given a continuous portfolio loss distribution function $F(x)$, the time t expected percentage cumulative loss on a given tranche can be written as:

$$EL_{(K_A,K_D)} = \frac{1}{K_D - K_A} \left(\int_{K_A}^{1} (x - K_A)\, dF(x) - \int_{K_D}^{1} (x - K_D)\, dF(x) \right) \tag{2.1}$$

Proof. (Ommitting the index (t_i))

$$\begin{aligned} EL_{(K_A,K_D)} &= \frac{1}{K_D - K_A} \sum_{n=1}^{N} \left(min(L_n, K_D) - K_A\right)^+ \, p_n \\ &= \frac{1}{K_D - K_A} \sum_{n=1}^{N} \left(\left(L_n \mathbf{1}_{\{L_n<K_D\}} + K_D \mathbf{1}_{\{L_n \geq K_D\}}\right) - K_A\right) \mathbf{1}_{\{min(L_n,K_D)>K_A\}} \, p_n \end{aligned}$$

$$
\begin{aligned}
EL_{(K_A,K_D)} &= \frac{1}{K_D-K_A}\sum_{n=1}^{N}\Big(\left(L_n\mathbf{1}_{\{L_n<K_D\}}-K_A\right)\mathbf{1}_{\{min(L_n,K_D)>K_A\}} \\
&\qquad\qquad + \left(K_D\mathbf{1}_{\{L_n\geq K_D\}}-K_A\right)\mathbf{1}_{\{min(L_n,K_D)>K_A\}}\Big)\,p_n \\
&= \frac{1}{K_D-K_A}\sum_{n=1}^{N}\Big(\left(L_n-K_A\right)\mathbf{1}_{\{L_n<K_D,L_n>K_A\}} \\
&\qquad\qquad + \left(K_D-K_A\right)\mathbf{1}_{\{K_D>K_A,L_n\geq K_D\}}\Big)\,p_n \\
&= \frac{1}{K_D-K_A}\left(\int_{K_A}^{K_D}(x-K_A)\,dF(x)+\int_{K_D}^{1}(K_D-K_A)\,dF(x)\right) \\
&= \frac{1}{K_D-K_A}\Big(\int_{K_A}^{1}(x-K_A)\,dF(x)-\int_{K_D}^{1}(x-K_A)\,dF(x) \\
&\qquad + \int_{K_D}^{1}(K_D-K_A)\,dF(x)\Big) \\
&= \frac{1}{K_D-K_A}\left(\int_{K_A}^{1}(x-K_A)\,dF(x)-\int_{K_D}^{1}(x-K_A-K_D+K_A)\,dF(x)\right) \\
&= \frac{1}{K_D-K_A}\left(\int_{K_A}^{1}(x-K_A)\,dF(x)-\int_{K_D}^{1}(x-K_D)\,dF(x)\right)
\end{aligned}
$$

■

2.2 The fair CDO premium

The fair price of a CDO tranche can be calculated using the same idea as for the pricing of a credit default swap. Namely, by setting the fair premium W such that the present values of the premium leg and the default leg are equal.

Let

$$0 \leq t_0 < ... < t_{m-1}$$

denote the premium payment dates.

The value of the premium leg PL of the tranche is the present value of all expected spread payments:

$$PL = \sum_{i=1}^{m} \Delta t_i \, W \, B(t_0, t_{i-1}) \left[1 - EL_{(K_A,K_D)}(t_{i-1})\right] \tag{2.2}$$

with $\Delta t_i = t_i - t_{i-1}$, $B(0, t_i)$ the discount factor and W the premium. We can see that the expected percentage loss EL reduces the amount of notional on which the premium W is paid. At beginning of the contract the premium is paid on 100% of the notional. Check for example that for a portfolio loss larger than K_D no premium will be paid anymore.

Similarly, the value of the default leg DL can be calculated as the expected value of the discounted default payments:

$$\begin{aligned} DL &= \int_{t_0}^{t_m} B(t_0, s) \, dEL_{(K_A,K_D)}(s) \\ &\approx \sum_{i=1}^{m} B(t_0, t_i) \left(EL_{(K_A,K_D)}(t_i) - EL_{(K_A,K_D)}(t_{i-1})\right) \end{aligned} \tag{2.3}$$

The fair price of the CDO tranche is then defined as the premium W^* such that

$$PL(W^*) - DL(W^*) = 0$$

and hence choosing the most compact (discretized) representation:

$$W^* = \frac{\sum_{i=1}^{m} B(t_0, t_i) \left(EL_{(K_A,K_D)}(t_i) - EL_{(K_A,K_D)}(t_{i-1})\right)}{\sum_{i=1}^{m} \Delta t_i \, B(t_0, t_{i-1}) \left[1 - EL_{(K_A,K_D)}(t_{i-1})\right]} \tag{2.4}$$

Equation 2.4 shows that as soon as we can calculate the expected loss $EL(t)$ for the tranche in question, the calculation of the premium is straight forward. Unfortunately the derivation of the distribution $F(x)$ of L, the loss on the reference portfolio, which is needed to calculate the tranche loss $EL(t)$ is not trivial. This is mainly due to the fact that we have to consider the dependence structure between obligors. Depending on the dependence between obligors the portfolio loss distribution can look completely different. The occurrence of disproportionally many defaults of different obligors in the reference portfolio will for example result in a heavy tailed loss distribution. The modelling of default dependence between obligors is therefore crucial when calculating loss distributions. This is already a daunting task, but with CDOs one has not only to consider joint defaults but also the timing of defaults, since the premium payment depends on the outstanding notional which is reduced during the lifetime of the contract if obligors default.

The aim of the following chapter is therefore to present the necessary theory to loss distribution modelling in a CDO pricing context.

Chapter 3

Loss distribution modelling - market practice

As shown in the previous chapter the probability distribution of the losses on the reference portfolio is a key input when pricing a CDO tranche. In the following the current market standard for the derivation of the loss distribution is presented. The idea is the following: Assume that the correlation of defaults on the reference portfolio is driven by common factors. Therefore, conditional on these common factors defaults are independent. To compute the unconditional loss distribution we just have to integrate over the common factors. Similar approaches have been followed by Li [17], Laurent and Gregory [16] and Andersen et al. [2].

In the next section we will first introduce the modelling framework and then explain various aspects of the model

3.1 The Gaussian factor copula model

In the well known firm-value models default occurs whenever a stochastic variable X_n (or a stochastic process in dynamic models) lies below a critical threshold K_n at the end of time period $[0, T]$. Specifically in the Merton model [19] default occurs when the value of the assets of a firm falls below the value of the firms' liabilities. In order to apply these models at portfolio level we require a multivariate version of a firm-value model. The factor copula model for CDO pricing was proposed by Li [17] and is heavily used by practitioners. In the factor copula model the critical variable X_n is interpreted as the default time of company n and it is assumed that X_n is exponentially distributed with parameter λ_n. Company n therefore only defaults by time

T if $X_n \leq T$. Since we assumed exponentially distributed default times, we can calculate the individual default probability of company n as:

$$p_n = e^{-\lambda_n T}$$

This will become extremely useful for calibration (see Section 3.3).

Let's further assume that the critical variable X_n depends on a single common factor Y, and X_n can therefore be written as:

$$X_n = \sqrt{\rho_n} Y + \sqrt{1 - \rho_n} \varepsilon_n \tag{3.1}$$

In a CDO the cash flows are functions of the whole random vector $\mathbf{X} = (X_1, ..., X_N)$. To evaluate a CDO, all we need is today's (risk neutral) joint distribution of the X_n's (assuming that there are N credits in the reference portfolio):

$$\mathbb{P}(X_1 < K_1, ..., X_N < K_N)$$

We can write the vector of critical variables $\mathbf{X}$ as:

$$\mathbf{X} = BY + \boldsymbol{\varepsilon} \tag{3.2}$$

where Y is a standard normally distributed random variable and ε_n for $n = 1, ..., N$ are independent univariate normally distributed random variables, which are also independent of Y.

Due to the stability of the normal distribution under convolution, the critical variables X_n follow a standard normal distribution as well. And the vector of critical variables $\mathbf{X}$ is multivariate normally distributed and hence $\mathbf{X}$ has the Gauss copula C_B^{Ga} for some equicorrelation B. As we will see, it is the flexibility of the copula functions that allows us to equip random variables with a Gaussian copula that are not normally distributed in their marginal distributions.

3.2 Copula functions

In order to understand why the vector of critical variables $\mathbf{X}$ in the factor copula model adopts the Gauss copula we first have to introduce the basic properties of copula functions. It will then become clear why a copula representation of the problem is extremely useful for dependency modelling

A copula is a multivariate distribution function with standard uniform margins.

Definition 3.1 ***Copula of* $\mathbf{X}$**. *The copula of $(X_1, ..., X_d)$ is the distribution function C of $(F_1(X_1), ..., F_d(X_d))$.*

Copulas very handy for modelling, because for any multivariate distribution, the univariate margins and the dependence structure can be separated. This property of copulas is stated in Sklar's theorem.

Theorem 1 ***Sklar's Theorem***. *Let F be a joint distribution function with margins $F_1, ..., F_d$. There exists a copula C such that for all $x_1, ..., x_d$ in $[-\infty, \infty]$.*

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$$

And conversely, if C is a copula and $F_1, ..., F_d$ are univariate distribution functions, then F is a multivariate distribution function with margins $F_1, ..., F_d$.

$$C(u_1, ..., u_d) = F(F_1^{\leftarrow}(u_1), ..., F_d^{\leftarrow}(u_d))$$

If the margins are continuous then C is unique. Otherwise C is uniquely determined on $\mathrm{Ran}F_1 \times ... \times \ \mathrm{Ran}F_d$.

The dependency modelling problem can therefore be divided in two parts: The first part is represented by the marginal distribution function of the random variables and the second part is the dependence structure between the random variables which is described by the copula function. From Sklar's Theorem it is also clear that any model for dependent defaults has an equivalent copula representation, although it may be difficult to write down the copula explicitly.

Also note that we are free to transform the X_n. This will not change the copula as long as all transformations are monotonously increasing.

Proposition 3.2 ***Invariance***. *C is invariant under strictly increasing transformations of the marginal distributions. If $T_1, ..., T_d$ are strictly increasing, then $(T_1(X_1), ..., T_d(X_d))$ has the same copula as $(X_1, ..., X_d)$.*

Turning back to the Gauss factor copula model introduced in section 3.1 we can now write the joint distribution of critical values using the copula terminology as:

$$F(X_1, ..., X_n) = C_B^{Ga}(F_1(X_1), ..., F_N(X_N))$$

where the $F_n(X_n)$ can be implied for each credit n from quoted market spreads.

And we can hence say that the critical values X_n adopt a Gaussian copula C_B^{Ga}:

$$C_B^{Ga}(\mathbf{u}) = \mathbf{\Phi_B}\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)\right)$$

(see also Schmidt and Ward [21]).

The Gaussian copula is popular for several reasons. It is very easy to draw random samples from it, correlated Gaussian random variables are well known, their dependence structure is well understood and it is to some degree analytically tractable.

To summarize, if we define the critical variables as done in Section 3.1 we firstly adopt a factor structure and secondly implicitly assume, as shown above, a Gaussian copula for the X_n. Therefore such a model is often called 'Gauss factor copula model'.

3.3 A note on calibration

The model is usually calibrated to observable market prices of credit default swaps. The default thresholds K_n are chosen such that they produce given individual default probabilities p_n implied from quoted credit default swap spreads.

$$K_n = \Phi^{-1}(p_n) \tag{3.3}$$

since:

$$p_n = \mathbb{P}\left[X_n \leq K_n\right] = \Phi(K_n)$$

If we didn't choose a factor structure for the X_n, we would have to estimate all the $\frac{1}{2}N(N-1)$ elements of the covariance matrix of the $X_1, ..., X_N$. Therefore to reduce the high dimensionality of the modelling problem a set of common factors is usually chosen which is assumed to drive the default dependency between firms. In the one factor model introduced above we therefore only have to estimate N numbers, (the ρ_n's), instead of $\frac{1}{2}N(N-1)$. The common factor can be interpreted as systematic risk factor affecting all obligors.

Remark 3.3.1 ***Conditional independence.*** *Due to the factor structure we chose for the critical variables X_n, defaults are conditionally independent. This works because as soon as we condition on the common factor the X_i only differ by their individual noise term ε_n which was defined as being independently distributed for all n and independent of Y. Therefore, after conditioning on the common factor Y the critical random variables $X_1, ..., X_N$ and therefore also defaults are independent.*

In the following we assume a homogeneous portfolio in the sense that all obligors have the same threshold $K_n = K$, that the notionals and recovery amounts of all obligors in the portfolio are the same, and that asset correlation is the same between all obligors $\rho_n = \rho$.

3.4 Conditional default probability

Conditioning on the common factor Y we can calculate the conditional default probability $p_n(y)$ for each obligor. This is the probability that the critical variable X_n falls below the threshold K, given that the common factor Y takes value y.

$$\begin{aligned} p_n(y) &= \mathbb{P}\left[X_n < K \,|\, Y = y\right] \\ &= \mathbb{P}\left[\sqrt{\rho}Y + \sqrt{1-\rho}\varepsilon_n < K \,|\, Y = y\right] \\ &= \mathbb{P}\left[\varepsilon_n < \frac{K - \sqrt{\rho}Y}{\sqrt{1-\rho}} \,|\, Y = y\right] \\ &= \Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \end{aligned} \tag{3.4}$$

Note that under the assumption of a large homogeneous portfolio the conditional individual default probabilities $p_n(y) = p\,(y)$ are the same for all obligors.

3.5 The loss distribution

Since defaults of different obligors in the portfolio are independent conditional on the realization of the common factor Y, and only two outcomes are possible (default or no default), the conditional probability of having exactly n defaults is given by the binomial distribution

$$\mathbb{P}\left[X = n \,|\, Y = y\right] = \binom{N}{n} p\left(y\right)^{n} \left(1 - p\left(y\right)\right)^{N-n} \tag{3.5}$$

Note that under the assumption of a homogeneous portfolio the probability of having exactly n out of N issuers that default is equal to the probability of the loss L being $L_n = \frac{n}{N}A(1-R)$.

To obtain the unconditional probability of having n defaults, we have to integrate over the common factor Y

$$\mathbb{P}\left[X = n\right] = \int_{-\infty}^{\infty} \mathbb{P}\left[X = n \,|\, Y = y\right] \phi(y)\, dy \tag{3.6}$$

Substituting 3.4 in 3.6 yields:

$$\mathbb{P}\left[X = n\right] = \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{n} \left(1 - \Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{N-n} \phi(y)\, dy. \tag{3.7}$$

Thus, the resulting distribution function of the defaults is:

$$\begin{aligned}\mathbb{P}\left[X \le m\right] &= \sum_{n=0}^{m} \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{n} \\ &\qquad \cdot \left(1 - \Phi\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{N-n} \phi(y)\, dy\end{aligned} \tag{3.8}$$

3.6 The large portfolio approximation for the one factor model

The calculation of the loss distribution in Equation 3.8 is computationally intensive, especially for large N. The large portfolio approximation proposed by Vasicek [25] and [26] is a convenient approximation method. Assume that the portfolio consists of very large number of obligors $N \to \infty$. Let X denote the fraction of the defaulted securities in the portfolio.

Hence, we want to calculate

$$\begin{aligned} F_N(x) &= \mathbb{P}\left[X \leq x\right] \\ &= \lim_{N\to\infty} \sum_{n=0}^{[Nx]} \int_{-\infty}^{\infty} \binom{N}{n} \left(\Phi\left(\frac{K-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^n \\ &\qquad \cdot \left(1-\Phi\left(\frac{K-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{N-n} \phi(y)\,dy. \end{aligned} \tag{3.9}$$

Then by substituting $s = \Phi\left(\frac{K-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)$ we get:

$$\begin{aligned} F_N(x) &= \mathbb{P}\left[X \leq x\right] \\ &= \lim_{N\to\infty} \sum_{n=0}^{[Nx]} \int_{-\infty}^{\infty} \binom{N}{n} s^n \left(1-s\right)^{N-n} d\Phi\left(\frac{\sqrt{(1-\rho)}\Phi^{-1}(s)-K}{\sqrt{\rho}}\right) \end{aligned}$$

And since

$$\lim_{N\to\infty} \sum_{n=0}^{[Nx]} \binom{N}{n} s^n \left(1-s\right)^{N-n} = \begin{cases} 0, & \text{if } x \leq s; \\ 1, & \text{if } x > s. \end{cases}$$

the cumulative distribution of losses of a large portfolio is given by

$$F_\infty(x) = \mathbb{P}\left[X \leq x\right] = \Phi\left(\frac{\sqrt{(1-\rho)}\Phi^{-1}(x)-K}{\sqrt{\rho}}\right) \tag{3.10}$$

Large portfolio results are convenient and as documented in Schönbucher [23] large portfolio limit distributions are often remarkably accurate approximations for finite-size portfolios especially in the upper tail. Given the uncertainty about the correct value for the asset correlation the small error generated by the large portfolio assumption is negligible.

Chapter 4

CDO pricing using the Gauss one factor copula model

4.1 The large portfolio approximation in the Gauss one factor copula model

In the previous chapter we provided all the tools needed to evaluate the CDO pricing formula. Recall the following formulas from Chapter 2:

The fair premium (Equation 2.4):

$$W^* = \frac{\sum_{i=1}^{m} B(t_0, t_i) \left(EL_{(K_A, K_D)}(t_i) - EL_{(K_A, K_D)}(t_{i-1}) \right)}{\sum_{i=1}^{m} \Delta t_i \, B(t_0, t_{i-1}) \left[1 - EL_{(K_A, K_D)}(t_{i-1}) \right]}$$

The expected loss (Equation 2.1):

$$EL_{(K_A, K_D)} = \frac{1}{K_D - K_A} \left(\int_{K_A}^{1} (x - K_A) \, dF(x) - \int_{K_D}^{1} (x - K_D) \, dF(x) \right)$$

And the loss distribution in the large portfolio approximation derived in the previous chapter (Equation 3.10):

$$F_\infty(x) = \mathbb{P}\left[X \leq x\right] \;\; = \;\; \Phi \left(\frac{\sqrt{(1-\rho)}\Phi^{-1}(x) - K}{\sqrt{\rho}} \right)$$

After calibration of the input parameters K and ρ it is therefore straightforward to calculate the CDO premium. The threshold K can be obtained by calibration of the individual default probabilities to observed market CDS spreads (see Arvantis and Gregory [3]). The equicorrelation can, for example, be implied from observed CDO tranche prices.

The Gauss one factor copula model is very convenient mainly because of the nice properties of the normal distribution. However, the assumption of a multivariate normal distributed vector of critical variables $\mathbf{X}$ may not be justified. The drawbacks of the multivariate normal assumption are discussed in the next section. The next section should also points out why the development of better models for CDO pricing is essential. Still, the Gauss one factor model serves as a benchmark model from where all extensions can be elaborated.

4.2 Drawbacks of the Gauss one factor copula approach

In a normal world dependence between random variables is measured by correlation. When investing in a CDO tranche, one is implicitly also trading correlation. To understand this, we first have to discuss what effect correlation has on the CDO tranche premia.

4.2.1 Correlation trading

Let's focus on the two tranches at the extreme, the equity tranche bearing the $0\% - 3\%$ losses and the supersenior tranche bearing the $30\% - 100\%$ of the losses in the reference portfolio. JPMorgan explains default correlation in a very intuitive example as analogously to a cat walking blindfolded through a room filled with mousetraps. If the cat has only one life (corresponding to the equity tranche), it would prefer the traps to be located in clusters (i.e. high correlation). The cat will lose its life whether it will hit only one trap or a whole cluster. At least with the traps in clusters there will be paths between them. Therefore the premium for an equity tranche on a portfolio with high correlation is lower than for low correlation. If the cat is a more traditional cartoon cat with nine lifes (senior tranche), it prefers the traps to be scattered evenly around the room (low correlation). It can afford to hit a few traps, but does not want to hit large clusters which would wipe out all its nine lifes. Therefore the premium of the senior tranche decreases with decreasing correlation.

CDO tranches are products allowing investors to take advantage of both, the views on default probability of obligors as well as on the correlation between defaults. Imagine for example a strategy where you sell protection on the equity tranche of a CDO and at the same time buy protection on a more senior tranche of the same CDO. With such a strategy you are 'long correlation'. If correlation increases, the premium for the equity tranche would decrease as it becomes less risky, whereas the premium for the more senior tranche would increase. Therefore such a strategy would pay if correlation would increase.

4.2.2 Implied correlation

Therefore, when premia of iTraxx CDO tranches are quoted in the market, they also incorporate a correlation calculation. The premium of a tranche is determined by bid and offer on the market. It is at the moment market standard to calculate the correlation value for a tranche using the Gauss one factor copula model. The implied compound correlation is the correlation value that produces a theoretical value equal to the market quote. Therefore, the standardisation of the market led to the price of correlation being set by the market. But just because the market is now pricing default correlation does not mean that default correlation is being priced correctly.

4.2.3 Correlation smile

In fact a theoretical issue arose when the market started to price default correlation. It turned out that for example on an iTraxx CDO the implied compound correlation is not the same across all the tranches. This phenomenon was called a 'correlation smile'. This is not consistent with the Gaussian one factor copula model where the correlation parameter must be the same for whatever tranche we look at. Different correlation parameters for different tranches in the Gaussian one factor copula model would mean that we assume different loss distributions for the same portfolio depending on which tranche we look at. This is complete nonsense. The conclusion is therefore that either the market is not pricing accurately or that the assumed model to calculate implied default correlation, the Gauss one factor copula model, is wrong.

Having a wrong assumption about default correlation values can be fatal when engaging in large scale correlation trades as the one described in section 4.2.1. Therefore, researchers and practitioners started to try to find a factor model which will match the market tranche premia more accurately than the

Gaussian one factor copula model.

Chapter 5

Extensions to the Gauss one factor copula model

The attractiveness of pricing CDOs using a Gauss one factor copula model is its simplicity and straightforward application, which only requires simple and fast numerical integration techniques. However, the model assumptions about the characteristics of the underlying portfolio are strong, and as discussed in Section 4.2 the fit to market data is not convincing.

In recent years researchers and practitioners came up with extensions to the Gauss one factor model, by relaxing one or several of the model assumptions. This chapter gives an overview on the most relevant, existing extensions. The chapter also serves as a basis for the in-depth treatment of three Gauss model extensions in Chapter 6.

5.1 Short review on existing extensions

5.1.1 More general distribution functions

In the Gauss one factor model a firm n defaults at time T if the random variable X_n is smaller than some threshold K. Where X_n is a function of a common systematic factor Y, and a firm idiosyncratic factor ε_n, assumed i.i.d. standard random normal variables.

Recall critical variable X_n:

$$X_n = \sqrt{\rho_n} Y + \sqrt{1 - \rho_n} \varepsilon_n$$

Then, the vector $\mathbf{X}$ of N critical variables is multivariate normal distributed (i.e. $\mathbf{X}$ adopts the Gauss copula):

$$\mathbf{X} \sim N_N(0, \Sigma)$$

However, there is no compelling reason for choosing normal random variables for the distribution of Y and $\varepsilon_1, ...\varepsilon_N$, and hence as a consequence for $X_1, ..., X_N$. The portfolio loss distribution is actually highly sensitive to the exact nature of the multivariate distribution of the critical variables $X_1, ..., X_N$ [10].

There are many alternatives to the Gauss copula. A popular family of distributions for modelling financial market returns is the family of multivariate normal mean-variance mixture models (see McNeil et al. [18]). When relaxing the assumption of multivariate normality it seems natural to look at this family which contains such distributions as the multivariate t and the hyperbolic.

Definition 5.1 ***Normal mean-variance mixtures.*** *The random vector* $\mathbf{X}$ *is said to have a (multivariate) normal mean-variance mixture distribution if*

$$\mathbf{X} \stackrel{d}{=} y\mathbf{m}(W) + \sqrt{W}\mathbf{Z} \ ,$$

where

1. $\mathbf{Z} \sim N_n(\mathbf{0}, \Sigma)$;
2. $W \geq 0$ is a non-negative, scalar-valued random variable which is independent of $\mathbf{Z}$;
3. $\mathbf{m} : [0, \infty) \to \mathbb{R}^n$ is a measurable function.

In this case we have that

$$\mathbf{X} \mid W = w \ \sim \ N_d(\mathbf{m}(w), w\Sigma),$$

and it is clear why such distributions are known as mean-variance mixtures of normals.

Example 2 ***Multivariate normal distribution.*** *In the special case where* $\mathbf{X} = \mathbf{Z}$ *and* $\mathbf{Z}$ *follows a linear factor model* $\mathbf{Z} = B\mathbf{Y} + \boldsymbol{\varepsilon}$ *with* $\mathbf{Y} \sim N_p(0, \Omega)$ *with* $p < N$ *and* $\varepsilon_n \sim N(0,1)$ *for* $\forall\, n$ *we are back to the the Gauss factor model.*

Extensions to the distribution assumption of the vector of critical variables X

Example 3 ***Multivariate t distribution.*** *If we take* W *to be a random variable with inverse gamma distribution* $W \sim Ig(\frac{1}{2}\nu, \frac{1}{2}\nu)$*, and* $\mathbf{X} = \sqrt{W}\mathbf{Z}$*, then* $\mathbf{X}$ *has a multivariate* t *distribution with* ν *degrees of freedom.*

Several authors have tested the assumption that the vector of critical variables $\mathbf{X}$ has a t Copula for CDO pricing (Galiano [11], Burtschell et al. [6]). The fit to market data was, however, not satisfying and the model was not able to produce the observed correlation smile.

Remark 5.1.1 *To directly sample from a known multivariate mixture distribution such as the normal or the t is especially nice for random number simulation and thus CDO pricing via Monte-Carlo simulation. This is because it is straightforward to simulate normal mixtures. For example for the multivariate t:*

1. *Generate* $\mathbf{Z} \sim N_n(\mathbf{0}, \Sigma)$.
2. *Generate* W *independently.*
3. *Set* $\mathbf{X} = \sqrt{W}\mathbf{Z}$

Example 4 ***Multivariate normal inverse Gaussian distribution.*** *If we take* W *to be a non-negative random variable with inverse Gaussian distribution* $W \sim IG(\alpha, \beta)$*, and*

$$\mathbf{X} \stackrel{d}{=} \mathbf{m}(W) + \sqrt{W}\mathbf{Z}$$

, then $\mathbf{X}$ *has a multivariate normal inverse Gaussian distribution (see McNeil et al. [18]).*

To our knowledge such a model has not been published. It could be quite promising since the multivariate normal inverse Gaussian, proofed to fit market equity return data quite well (McNeil et al. [18]). As in the multivariate t case the sampling from the normal inverse Gaussian copula is very simple. For the calculation of the large portfolio approximation in a normal inverse Gaussian copula model see Appendix A.

Extensions to the distribution assumptions of the systematic factor Y and idiosyncratic factors ε_n

Another approach that has been tried was to change the distributional assumptions not on $\mathbf{X}$ but on the systematic factor Y and the idiosyncratic noise term ε_n. Recall that in the Gauss one factor copula model we assume for both random variables a normal distribution.

Example 5 ***Double t model.*** *Hull and White [14] assumed for both, the systematic factor and the idiosyncratic noise term a student t distribution. Unfortunately the t distribution is not stable under convolution, hence the critical variables X_n are not student t distributed. Therefore, this model requires rather time consuming numerical calculations and is usually too slow for practitioners. Note, that the vector of critical variables $\mathbf{X}$ is not multivariate t distributed, nor does it adopt the t copula! The multivariate distribution of the vector is unknown as well as is the copula. It would therefore be misleading to speak of a t copula model in such a set-up.*

Even if the t distribution were stable under convolution the vector of critical variables $\mathbf{X}$ would not have the t copula. This is due to the fact that *i.i.d.* ε_n are assumed, however, for a multivariate normal mixture distribution the marginals cannot be independent (see Lemma 5.2).

Lemma 5.2 *Let (X_1, X_2) have a normal mixture distribution with $\mathbb{E}(W) < \infty$ so that $cov(X_1, X_2) = 0$. Then X_1 and X_2 are independent if and only if W is almost surely constant, i.e. (X_1, X_2) are normally distributed.*

For the proof see [18].

Example 6 ***Double normal inverse Gaussian model*** *Another similar approach that has been tried, is to assume a normal inverse Gaussian distribution for both, the systematic factor Y and the idiosyncratic noise term ε_n (Kalemanova et al [15]). The advantage of this model is that the normal inverse Gaussian distribution is stable under convolution. That means in this model also the critical variables X_n are normal inverse Gaussian distributed. However, because we assume i.i.d. idiosyncratic noise ε_n the vector of critical variables $\mathbf{X}$ is again not multivariate normal inverse Gaussian distributed and does not have the normal inverse Gaussian copula (due to Lemma 5.2) and is therefore not identical to the normal inverse Gaussian copula model in Example 4!*

5.1.2 Stochastic correlation and further extensions

Introducing stochastic default correlation

In the Gauss one factor copula model default correlations are assumed to be constant through time, the same for all firms and independent of the firms' default probabilities. Andersen and Sidenius [1] consider a model with stochastic default correlation, allowing default correlation to be higher in bear markets than in bull markets. Burtschell et al [6] have relaxed these assumptions to allow for stochastic correlations. Both models showed an improved fit to market data compared to the Gauss one factor model.

Further extensions

Further extensions of the Gauss one factor copula are the assumption of a finite heterogenous portfolio instead of the homogenous large portfolio approximation, not a one factor but a multi factor model (see Schönbucher [23]). Andersen and Sidenius [1] relaxed the assumption of constant recovery rates by introducing stochastic recovery rates. A multifactor model for correlation introducing group structure has also been tried (Burtschell et al. [6]). To introduce a group structure is intuitively appealing, because to group obligors according to industry sector with high within group correlation but low between group correlation makes sense. However, none of these models erased the correlation smile and the multifactor models are computationally intensive.

Chapter 6

Comparison of four CDO pricing models: The Gauss model, two known extensions and one new extension

In this chapter four different large portfolio models for CDO pricing are presented in detail. The Gauss copula model serves as a benchmark model as it is the standard model for practitioners.

In the extension to the Gauss copula model a) the distributional assumptions are relaxed, and b) stochastic factor loadings are introduced. In all the models the one factor setting is retained as well as the large portfolio approximation assumption introduced in Section 3.6. The extensions to the Gauss model considered here are: the double normal inverse Gaussian (double NIG) model introduced by Kalemanova et al. [15] and the Gauss one factor model with stochastic factor loadings introduced by Andersen and Sidenius [1]. As a third model to be compared to the Gauss benchmark model a new model is introduced which unites the double NIG extension with stochastic factor loadings. We specifically chose these extensions because they provide the most promising results in terms of fit to market data and computational efficiency.

6.1 The Gauss one factor copula model

For completeness the model set-up and the main findings of the Gauss one factor copula model are briefly reviewed here. For more details see Chapter 3.

6.1.1 Model set-up in the Gauss one factor model

In the Gauss one factor copula model the critical variables X_n are given by:

$$X_n = \sqrt{\rho} Y + \sqrt{1-\rho}\varepsilon_n$$

where

- ρ constant equicorrelation;
- $Y \sim N(0,1)$;
- $\varepsilon_n, n = 1, ..., N$ are i.i.d. standard normal random variables, independent of Y;
- hence $X_n \sim N(0,1)$ and $\mathbf{X} \sim N_N(0, \Sigma)$ (due to stability under convolution and Lemma 5.2).

6.1.2 Large portfolio approximation in the Gauss one factor copula model

As derived in Section 3.6 the cumulative loss distribution under the large portfolio assumption is given by:

$$F_\infty(x) = \mathbb{P}\left[X \leq x\right] \quad = \quad \Phi\left(\frac{\sqrt{(1-\rho)}\Phi^{-1}(x) - K}{\sqrt{\rho}}\right)$$

And hence we can calculate the tranche expected losses in Equation 2.1 and CDO premiums in Equation 2.4 as explained in Section 4.1 easily using some numerical integration.

6.2 The double normal inverse Gaussian one factor copula model

The family of normal inverse Gaussian distributions is a special case of the group of generalized hyperbolic distributions (Barndorff-Nielsen [4]). Due to their specific characteristics the family of normal inverse Gaussian distributions are very interesting for applications in finance. They are generally flexible four parameter distributions that can produce fat tails and skewness. They are stable under convolution under certain conditions and the cumulative distribution function, density and inverse distribution function can still be computed sufficiently fast. Also normal inverse Gaussian models have been popular in financial applications as they proofed to provide a good fit to financial return data (McNeil et al. [18] and Banrdorff-Nielsen [5]).

The double normal inverse Gaussian one factor model (double NIG) was applied recently to CDO pricing problems by Kalemanova et al. [15] and produced a good fit to market data. In this thesis the approach adopted by Kalemanova et al. [15] will be followed.

6.2.1 Model set-up in the double NIG model

In the double NIG one factor copula model the critical variables are given by:

$$X_n = \sqrt{\rho} Y + \sqrt{1-\rho}\varepsilon_n$$

where

- ρ constant equicorrelation;
- $Y \sim NIG$;
- $\varepsilon_n, n = 1, ..., N$ are i.i.d. normal inverse Gaussian random variables, independent of Y;
- hence, because of stability under convolution $X_n \sim F_{NIG}(0, 1)$.

Note that even though the X_n are normal inverse Gaussian distributed, the vector of critical variables $\mathbf{X}$ has *not* a multivariate normal inverse Gaussian distribution! As stated in Lemma 5.2 a multivariate normal inverse Gaussian distribution cannot have independent marginals but only uncorrelated marginals. It is therefore slightly misleading to name this model 'the normal inverse Gaussian copula model' (Kalemanova et al. [15]). Especially, because a true normal inverse Gaussian copula model can be set-up following Section 5.1.1 and Appendix A. We therefore suggest the model (as set-up in this Section) should be called the double normal inverse Gaussian copula model (double NIG) which would be inline with the analogue set up of the double t model by Hull and White [14].

6.2.2 Definition and properties of the NIG distribution

The normal inverse Gaussian distribution is a mixture of a normal and an inverse Gaussian distribution.

A non-negative random variable Y has inverse Gaussian distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density is of the form:

$$f_{IG}(y;\alpha,\beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}}\, y^{\frac{3}{2}}\, e^{-\frac{(\alpha-\beta y)^2}{2\beta y}} & \text{if } y > 0; \\ 0 & \text{if } y \leq 0. \end{cases}$$

A random variable X follows a normal inverse Gaussian distribution with parameters α, β, μ and δ if:

$$\begin{aligned} X \mid Y = y &\sim N(\mu + \beta y, y) \\ Y &\sim IG(\delta\lambda, \lambda^2)\ with\ \lambda := \sqrt{\alpha^2 - \beta^2} \end{aligned}$$

with parameters satisfying the following conditions: $0 \leq \mid \beta \mid < \alpha$ and $\delta > 0$. We write $X \sim NIG(\alpha, \beta, \mu, \delta)$ and denote the density and distribution functions by $f_{NIG}(x;\alpha,\beta,\mu,\delta)$ and $F_{NIG}(x;\alpha,\beta,\mu,\delta)$.

The density of a random variable $X \sim NIG(\alpha, \beta, \mu, \delta)$ is:

$$f_{NIG}(x;\alpha,\beta,\mu,\delta) = \frac{\delta\alpha\ exp\ (\delta\lambda + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}}\ K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) \quad (6.1)$$

where $K_1 := \frac{1}{2}\int_0^\infty exp\left(-\frac{1}{2}\ w(t + t^{-1})\right) dt$ is the modified Bessel function of the third kind.

The density depends on four parameters:

- $\alpha > 0$ determines the shape,
- β with $0 \leq \mid \beta \mid < \alpha$ the skewness,
- μ the location, and
- $\delta > 0$ is a scaling parameter.

For illustration see Fig. 6.1.

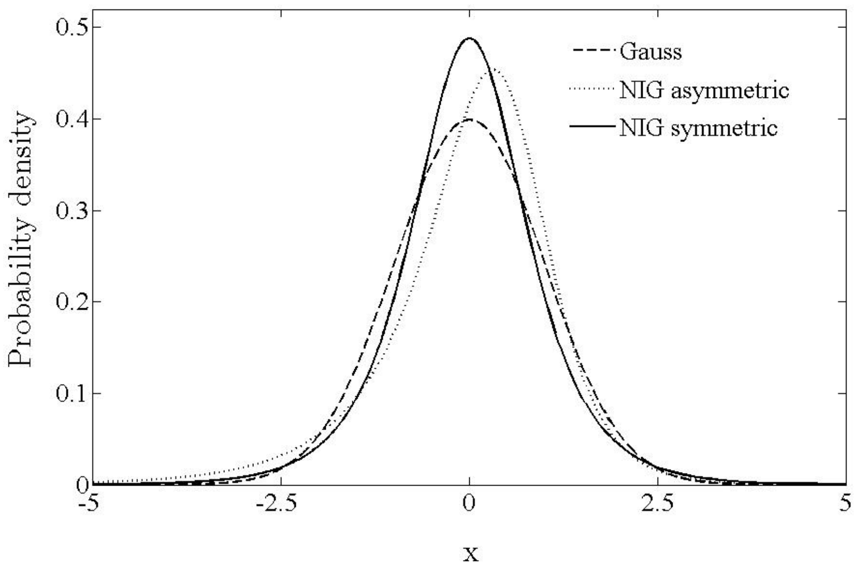

Figure 6.1: Normal inverse Gaussian distribution (NIG) in comparison to the normal distribution. The shape and skewness parameters of the asymmetric NIG are: $\alpha = 1.2$ and $\beta = -0.5$; and of the symmetric NIG: $\alpha = 1.2$ and $\beta = 0$.

The mean and variance of a random variable $X \sim NIG(\alpha, \beta, \mu, \delta)$ are given by:

$$\mathbb{E}\left[X\right] = \mu + \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \qquad\qquad \mathbb{V}\left[X\right] = \delta \frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}^3} \tag{6.2}$$

The main property of the NIG distribution class is the scaling property

$$X \sim NIG(\alpha, \beta, \mu, \delta) \Rightarrow c\,X \sim NIG\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\,\mu, c\,\delta\right)$$

and the closure under convolution for independent random variables X and Y

$$\begin{array}{rrcl} & X & \sim & NIG(\alpha, \beta, \mu_1, \delta_1) \\ & Y & \sim & NIG(\alpha, \beta, \mu_2, \delta_2) \\ \Rightarrow & X + Y & \sim & NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2) \end{array}$$

Recall that in the double NIG one factor copula model the critical variables are given by:

$$X_n = \sqrt{\rho}Y + \sqrt{1-\rho}\varepsilon_n$$

where $Y, \varepsilon_n, n = 1, ..., N$ are independent normal inverse Gaussian variables with the following parameters:

$$\begin{aligned} Y &\sim NIG\left(\alpha, \beta, -\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, \alpha\right) \\ \varepsilon_n &\sim NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha\right). \end{aligned}$$

The parameters are chosen such that we get zero expected value and identical variance for the distribution of Y, ε_n and X_n. For non-zero β we get a skewed distribution.

Then by applying the scaling property and stability under convolution the critical variables X_n are NIG distributed with the following parameters

$$X_n \sim NIG\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, \frac{\alpha}{\sqrt{\rho}}\right). \tag{6.3}$$

To simplify notation we denote

$$F_{NIG}\left(x;\ s\alpha, s\beta, -s\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, s\alpha\right)$$

with

$$F_{NIG(s)}(x)$$

.

6.2.3 Large portfolio approximation in the double NIG copula model

Because of the stability under convolution the default thresholds K are easy and relatively fast to compute:

$$K = F^{-1}_{NIG\left(\frac{1}{\sqrt{\rho}}\right)}(p).$$

Note, that in contrast to the Gauss model, in the double NIG model one has to be very careful with the parameters (denoted by s) of the NIG distribution. The parameters differ depending on whether the critical variable X_n, the factor Y or the idiosyncratic term ε_n is considered.

Using exactly analogous steps as for the Gauss one factor copula model in Section 3.4 the default probability of each obligor in the portfolio is given by:

$$p_n(y) \quad = \quad F_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}\left(\frac{K - \sqrt{\rho}y}{\sqrt{1-\rho}}\right),$$

and following the arguments in Section 3.6 the loss distribution under the large homogeneous portfolio approximation is given by

$$F_\infty(x) = \mathbb{P}\left[X \leq x\right] \quad = \quad F_{NIG(1)}\left(\frac{\sqrt{(1-\rho)}F^{-1}_{NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)}(x) - K}{\sqrt{\rho}}\right).$$

6.2.4 Efficient implementation of the NIG distribution

As shown before the density of the NIG distribution is rather complicated as it involves a Bessel function. For CDO pricing an efficient implementation of the NIG distribution is essential. Fortunately this is possible, and speed is therefore not an issue when discussing pros and cons of the double NIG model. An efficient implementation of the NIG distribution, density and inverse distribution can be found for Matlab at

`http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=6050&objectType=file`

The code was provided by Ralf Werner from Risklab, Germany. For the calculation of the inverse distribution it is computationally much more efficient to table the distribution function F_{NIG} on a fine grid around the mean. This table can then be used to find the inverse.

6.3 The Gauss one factor copula model with stochastic factor loading

As is shown for example by Andersen and Sidenius [1] the market loss distribution is more 'kinked' than the loss distribution produced by the Gauss model with empirical correlations. The market loss distribution has a fatter upper tail and relatively low probability to produce small losses. It was therefore obvious to try a student t copula model which produces a fat tail (see e.g. Frey and McNeil [9]). However, it is difficult to do so without increasing the probability of generating zero losses too much.

The introduction of random factor loadings in the Gauss model is a more gentle way of generating a 'kinked' loss distribution. The problem of producing zero losses is avoided.

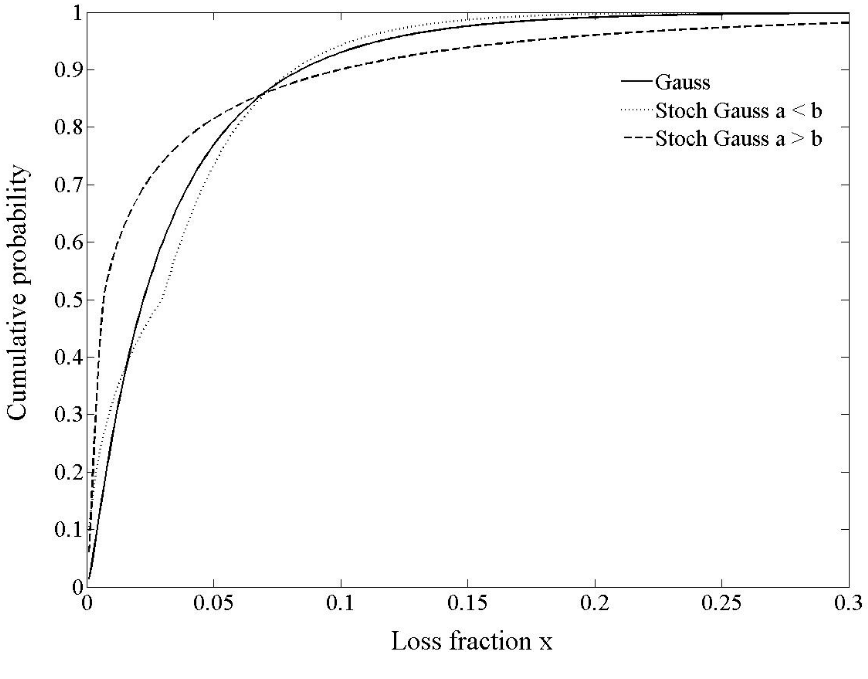

Figure 6.2: Lossdistribution with random factor loadings model (Stoch Gauss) in comparison to the lossdistribution produced by the Gauss model.

In the random factor loadings model the factor loadings are made functions of the systematic factors themselves. This model was introduced by Andersen and Sidenius [1]. One of the nice properties of this model is that the model extension has an intuitively clear justification. Assume that the systematic factor represents the state of the market, with its value being low in bad times and higher in good times. In the random factor loadings model we can then mimic the well known empirical effect that equity (and thereby asset) correlation is high in bull markets and lower in bear markets.

6.3.1 Model set-up with random factor loadings

In the Gauss one factor copula model with random factor loadings the critical variables X_n are given by:

$$X_n = a(Y)Y + v\,\varepsilon_n + m$$

where as in the Gaussian one factor model:

- $Y \sim N(0,1)$;
- $\varepsilon_n, n = 1, ..., N$ are i.i.d. standard normal random variables, independent of Y;
- v and m are set such that X_n have zero mean and unit variance.

but where

$$a(Y) = \begin{cases} \sqrt{a} & \text{if } Y \leq \theta; \\ \sqrt{b} & \text{if } Y > \theta. \end{cases}$$

with a, b positive constants and $\theta \in \mathbb{R}$ and hence the random factor loading is introduced by a simple two-point distribution. Continuing in line with the intuition started above the regime switching model is introduced via a 2-point distribution, where the loading takes value $\sqrt{a}$ with probability $\Phi(\theta)$ and value $\sqrt{b}$ with probability $1 - \Phi(\theta)$.

If $a > b$ the factor loadings decrease in Y and thus, asset values couple more strongly to the economy in bad times than in good times. To understand why such a model could produce a base correlation skew consider the view of a senior tranche investor. This investor will only experience losses on his position if several names default together, that is in scenarios where the systematic factor Y is likely to take on low values. If Y is low, the factor loadings will be high, making it appear to senior investors that correlations are high. For the equity investor on the other hand, who is likely to experience losses even in scenarios where Y is not low, the effective factor loading will appear as a weighted average between $\sqrt{a}$ and $\sqrt{b}$. To the equity investor the world will therefore look as if correlations are of average magnitude. Clearly if $a = b$ we are back to the constant factor loading of the Gaussian case.

Calculation of parameters v and m

In order to get zero mean unit variance for X_n we need v and m to be:

$$v = \sqrt{1 - \mathbb{V}(a(Y)Y)}$$

$$m = -\mathbb{E}\left[a(Y)Y\right]$$

that leaves to calculate:

$$\begin{aligned}
\mathbb{E}\left[a(Y)Y\right] &= \mathbb{E}\left[\sqrt{a}1_{Y\le\theta}Y + \sqrt{b}1_{Y>\theta}Y\right] \\
&= (-\sqrt{a}\varphi(\theta) + \sqrt{b}\varphi(\theta)) \qquad (6.4) \\
\mathbb{E}\left[a(Y)^2Y^2\right] &= \mathbb{E}\left[a1_{Y\le\theta}Y^2 + b1_{Y>\theta}Y^2\right] \\
&= a\left(\Phi(\theta) - \theta\varphi(\theta)\right) + b\left(\theta\varphi(\theta) + 1 - \Phi(\theta)\right) \qquad (6.5)
\end{aligned}$$

from where the result for, m and v immediately follow.

$$\begin{aligned}
m &= -(-\sqrt{a}\varphi(\theta) + \sqrt{b}\varphi(\theta)) \qquad (6.6) \\
v &= \sqrt{1 - \mathbb{V}(a(Y)Y)} \qquad (6.7)
\end{aligned}$$

with

$$\mathbb{V}(a(Y)Y) = a\left(\Phi(\theta) - \theta\varphi(\theta)\right) + b\left(\theta\varphi(\theta) + 1 - \Phi(\theta)\right) - \left(-\sqrt{a}\varphi(\theta) + \sqrt{b}\varphi(\theta)\right)^2 \qquad (6.8)$$

Note that in 6.4 and in 6.5 above we used the following lemma:

Lemma 6.1 *For a standardized Gaussian variable x and arbitrary constants c and d, we have*

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{c<x\le d}\, x\right] &= \mathbf{1}_{d\ge c}\left(\varphi(c) - \varphi(d)\right); \qquad (6.9) \\
\mathbb{E}\left[\mathbf{1}_{c<x\le d}\, x^2\right] &= \mathbf{1}_{d\ge c}\left(\Phi(d) - \Phi(c)\right) + \mathbf{1}_{d\ge c}\left(c\varphi(c) - d\varphi(d)\right). \qquad (6.10)
\end{aligned}$$

In particular

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{x \leq d}\, x\right] &= -\varphi(d); \\
\mathbb{E}\left[\mathbf{1}_{x > c}\, x\right] &= \varphi(c); \\
\mathbb{E}\left[\mathbf{1}_{x \leq d}\, x^2\right] &= \Phi(d) - d\varphi(d); \\
\mathbb{E}\left[\mathbf{1}_{x > c}\, x^2\right] &= c\varphi(c) + (1 - \Phi(c))\,.
\end{aligned}
$$

Proof. If $d < c$, the expectations in 6.9 and 6.10 are obviously zero. For $d \geq c$ we have,

$$
\mathbb{E}\left[\mathbf{1}_{c < x \leq d}\, x\right] = \mathbf{1}_{d>c} \int_c^d x\varphi(x)dx = -\mathbf{1}_{d>c} \int_c^d \varphi'(x)dx = \mathbf{1}_{d>c}\left(\varphi(c) - \varphi(d)\right), \tag{6.11}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{c < x \leq d}\, x^2\right] &= \mathbf{1}_{d>c} \int_c^d x^2\varphi(x)dx = \mathbf{1}_{d>c} \int_c^d \left(\varphi''(x) + \varphi(x)\right)dx \\
&= \mathbf{1}_{d>c}\left(\Phi(d) - \Phi(c)\right) + \mathbf{1}_{d>c}\left(c\varphi(c) - d\varphi(d)\right). \qquad (6.12)
\end{aligned}
$$

■

Distribution of X_n

If $a \neq b$ then X_n are in general not normally distributed anymore, and $\mathbf{X}$ will *not* be multivariate normal distributed. Therefore, the default barriers K cannot be easily calculated as in the Gaussian case by

$$
K = \Phi^{-1}(p)
$$

In the stochastic factor loading model the individual default probability is given by

$$\begin{aligned}
p &= \mathbb{P}(\tau \leq T) = \mathbb{P}(X_n \leq yK) \\
&= \mathbb{P}\left(\sqrt{a}\mathbf{1}_{Y\leq\theta}Y + \sqrt{b}\mathbf{1}_{Y>\theta}Y + \varepsilon_n v + m \leq K\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\varepsilon_n \leq \frac{K - \left(\sqrt{a}\mathbf{1}_{Y\leq\theta}Y + \sqrt{b}\mathbf{1}_{Y>\theta}Y\right) - m}{v} \mid Y\right)\right] \\
&= \int_{-\infty}^{\theta} \Phi\left(\frac{K - \sqrt{a}Y - m}{v}\right)\varphi(Y)dY + \int_{-\infty}^{\theta} \Phi\left(\frac{K - \sqrt{b}Y - m}{v}\right)\varphi(Y)dY
\end{aligned} \tag{6.13}$$

This expression allows to numerically find the default trigger levels K from given default probabilities.

6.3.2 Large portfolio approximation in the Gauss one factor copula model with random factor loadings

Consider the individual conditional default probability (compare to the Gaussian case in equation 3.4).

$$\begin{aligned}
p\,(y) &= \mathbb{P}\,(X_1 \leq K \mid Y) \\
&= \mathbb{P}\left(\varepsilon_1 \leq \frac{K - a(Y)Y - m}{v}\right) \\
&= \Phi\left(\frac{K - 1_{Y\leq\theta}\sqrt{a}Y - 1_{Y>\theta}\sqrt{b}Y - m}{v}\right).
\end{aligned} \tag{6.14}$$

Then the large portfolio approximation justifies that conditional on $Y = y$ the loss X, the fraction of defaulted securities in the portfolio, is given by $p\,(y)$.

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(X \geq x) &= \mathbb{P}\,(p\,(y) \geq x) \\
&= \mathbb{P}\left(\frac{K - a(Y)Y - m}{v} \geq \Phi^{-1}(x)\right) \\
&= \mathbb{P}\left(a(Y)Y \leq K - v\,\Phi^{-1}(x) - m\right)
\end{aligned}$$

denote by $\Omega(x) := K - v\,\Phi^{-1}(x) - m$

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(X \geq x) &= \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(\sqrt{b}Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= \Phi\left(\min\left(\frac{\Omega(x)}{\sqrt{a}}\,,\, \theta\right)\right) + 1_{\frac{\Omega(x)}{\sqrt{b}} > \theta}\left(\Phi\left(\frac{\Omega(x)}{\sqrt{b}}\right) - \Phi(\theta)\right).
\end{aligned}
\tag{6.15}
$$

Proof.

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(X \geq x) &= \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(\sqrt{b}Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) \\
&\quad + \mathbb{P}(\sqrt{b}Y \leq \Omega(x))\, \mathbb{P}\left(Y > \theta\right) \mathbf{1}_{\left\{Y \leq \frac{\Omega(x)}{\sqrt{b}}, Y > \theta\right\}} \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) \\
&\quad + \mathbb{P}(\sqrt{b}Y \leq \Omega(x))\ \left(1 - \mathbb{P}\left(Y \leq \theta\right)\right) \mathbf{1}_{\left\{Y \leq \frac{\Omega(x)}{\sqrt{b}}, Y > \theta\right\}} \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \Big(\mathbb{P}(\sqrt{b}Y \leq \Omega(x)) \\
&\quad - \mathbb{P}(\sqrt{b}Y \leq \Omega(x))\mathbb{P}\left(Y \leq \theta\right)\Big) \mathbf{1}_{\left\{Y \leq \frac{\Omega(x)}{\sqrt{b}}, Y > \theta\right\}} \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \Big(\mathbb{P}(\sqrt{b}Y \leq \Omega(x)) \\
&\quad - \mathbb{P}\left(\sqrt{b}Y \leq \Omega(x) \cap Y \leq \theta\right)\Big) \mathbf{1}_{\left\{Y \leq \frac{\Omega(x)}{\sqrt{b}}, Y > \theta\right\}} \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) \\
&\quad + \left(\mathbb{P}(\sqrt{b}Y \leq \Omega(x))\ -\ \mathbb{P}\left(Y \leq \theta\right)\right) \mathbf{1}_{\left\{Y \leq \frac{\Omega(x)}{\sqrt{b}}, Y > \theta\right\}} \\
&= \Phi\left(\min\left(\frac{\Omega(x)}{\sqrt{a}}\,,\, \theta\right)\right) + 1_{\frac{\Omega(x)}{\sqrt{b}} > \theta}\left(\Phi\left(\frac{\Omega(x)}{\sqrt{b}}\right) - \Phi(\theta)\right)
\end{aligned}
$$

$\blacksquare$

Hence the cumulative loss distribution in the random factor loadings model is given by

$$
\begin{aligned}
F_\infty(x) &= 1 - \lim_{N\to\infty} \mathbb{P}(X \geq x) \\
&= 1 - \left[\Phi\left(\min\left(\frac{\Omega(x)}{\sqrt{a}}\ ,\ \theta \right)\right) + 1_{\frac{\Omega(x)}{\sqrt{b}} > \theta} \left(\Phi\left(\frac{\Omega(x)}{\sqrt{b}} \right) - \Phi(\theta) \right) \right]
\end{aligned}
$$

which is numerically still very efficient to implement.

6.4 The double NIG one factor copula model with stochastic factor loading

As shown by Andersen and Sidenius [1] for the random factor loadings and by Kalemanova et al. [15] for the double NIG model these two models seem to fit market data quite well. It was therefore a natural extension to see if a combination of the two in one model would provide an even better fit. In the following such a double NIG one factor model with stochastic factor loadings is developed.

6.4.1 Model set-up of the double NIG model with random factor loadings

In the double NIG one factor copula model with random factor loadings the critical variables X_n are given by:

$$X_n = a(Y)Y + v\,\varepsilon_n + m$$

where as in the double NIG one factor model:

- $Y \sim NIG\left(\alpha, \beta, -\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, \alpha\right)$;
- $\varepsilon_n \sim NIG\left(c\,\alpha, c\,\beta, -c\,\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, c\,\alpha\right)$ are i.i.d. normal inverse Gaussian random variables, independent of Y (for the specification of the constant c see below);

- v and m are set such that the X_n have zero mean and the same variance as Y and ε_n.

and where as in the Gauss model the random factor loading is introduced via a two-point ('regime switching') distribution:

$$a(Y) = \begin{cases} \sqrt{a} & \text{if } Y \leq \theta; \\ \sqrt{b} & \text{if } Y > \theta. \end{cases}$$

with a, b positive constants and $\theta \in \mathbb{R}$.

To simplify notation we denote as before

$$F_{NIG}\left(x;\ s\alpha, s\beta, -s\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, s\alpha\right)$$

by

$$F_{NIG(s)}(x)$$

and

$$f_{NIG(s)}(x)$$

respectively.

Calculation of parameters v and m

v and m are set such that X_n have zero mean and the same variance as Y and ε_n.

For a random variable as defined above:

$$\varepsilon_n \sim NIG\left(c\,\alpha, c\,\beta, -c\,\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, c\,\alpha\right) \tag{6.16}$$

the mean and variance as given by Equation 6.2 are:

$$\mathbb{E}\left[\varepsilon_n\right] = 0 \qquad\qquad \mathbb{V}\left[\varepsilon_n\right] = \frac{\alpha^3}{\sqrt{\alpha^2-\beta^2}^3}$$

Hence, mean and variance as defined in Equation 6.16 are independent of the constant c.

Therefore

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_n\right] &= \mathbb{E}\left[Y\right] = 0 \\
\mathbb{V}\left[\varepsilon_n\right] &= \mathbb{V}\left[Y\right] = \frac{\alpha^3}{\sqrt{\alpha^2-\beta^2}^3}
\end{aligned}$$

And we also require for X_n:

$$\mathbb{E}\left[X_n\right] = 0 \qquad\qquad \mathbb{V}\left[X_n\right] = \frac{\alpha^3}{\sqrt{\alpha^2-\beta^2}^3}$$

which allows the calculation of m and v. Indeed, for m we have

$$0 \;=\; \mathbb{E}\left[X_n\right] \;=\; \mathbb{E}\left[a(Y)Y\right] + v\,\mathbb{E}\left[\varepsilon_n\right] + m$$

that rewritten is

$$\begin{aligned}
m &= -\mathbb{E}\left[a(Y)Y\right] \\
&= -\mathbb{E}\left[\sqrt{a}\mathbf{1}_{Y\le\theta}\,Y + \sqrt{b}\mathbf{1}_{Y>\theta}\,Y\right] \\
&= -\left(\sqrt{a}\int_{-\infty}^{\theta} Y dF_{NIG(1)}(Y) + \sqrt{b}\int_{\theta}^{\infty} Y dF_{NIG(1)}(Y)\right).
\end{aligned}$$

For v, by denoting $\gamma = \frac{\alpha^3}{\sqrt{\alpha^2-\beta^2}^3}$, we have

$$\begin{aligned}
\gamma &= \mathbb{V}\left[X_n\right] = \mathbb{V}\left[a(Y)Y\right] + v^2\mathbb{V}\left[\varepsilon_n\right] + 2Cov(a(Y)Y,\varepsilon_n)v \\
&= \mathbb{V}\left[a(Y)Y\right] + v^2\,\gamma
\end{aligned}$$

that rewritten is

$$v = \sqrt{1 - \frac{\mathbb{V}\left[a(Y)Y\right]}{\gamma}},$$

with

$$
\begin{aligned}
\mathbb{V}\left[a(Y)Y\right] &= \mathbb{E}\left[a(Y)^2Y^2\right] - \mathbb{E}\left[a(Y)Y\right]^2 \\
&= \mathbb{E}\left[a\mathbf{1}_{Y\leq\theta}\,Y^2 + b\mathbf{1}_{Y>\theta}\,Y^2\right] - \mathbb{E}\left[\sqrt{a}\mathbf{1}_{Y\leq\theta}\,Y + \sqrt{b}\mathbf{1}_{Y>\theta}\,Y\right]^2 \\
&= a\int_{-\infty}^{\theta} Y^2 dF_{NIG(1)}(Y) + b\int_{\theta}^{\infty} Y^2 dF_{NIG(1)}(Y) \\
&\quad -\left(\sqrt{a}\int_{-\infty}^{\theta} Y dF_{NIG(1)}(Y) + \sqrt{b}\int_{\theta}^{\infty} Y dF_{NIG(1)}(Y)\right)^2.
\end{aligned}
$$

It is trivial to show that in the case when $a = b$ the parameter v is given as in the double NIG model without factor loadings by $\sqrt{1-a}$.

Note that for v and m we don't get so nice expressions as in the Gaussian model, because Lemma 6.1 does not hold for the normal inverse Gaussian distribution. For the Lemma to hold we would require that $x\varphi(x) = -\varphi'(x)$ which does not hold in the normal inverse Gaussian case. (This is already obvious without calculation when looking at the form of f_{NIG} in Equation 6.1).

Distribution of X_n

The model is set-up such that when $a = b$ we are back to the double NIG model where the X_n are normal inverse Gaussian distributed due to the stability under convolution. If $a \neq b$ then X_n are in general not normal inverse Gaussian distributed anymore. Therefore, the default barriers K cannot be easily calculated as in the double NIG model by

$$K = F^{-1}_{NIG\left(\frac{1}{\sqrt{V}}\right)}(p).$$

In the double NIG model with stochastic factor loading model the individual default probability is given by

$$
\begin{aligned}
p &= \mathbb{P}(\tau \leq T) = \mathbb{P}(X_n \leq yK) \\
&= \mathbb{P}\left(\sqrt{a}\mathbf{1}_{Y\leq\theta}Y + \sqrt{b}\mathbf{1}_{Y>\theta}Y + \varepsilon_n v + m \leq K\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\varepsilon_n \leq \frac{K - \left(\sqrt{a}\mathbf{1}_{Y\leq\theta}Y + \sqrt{b}\mathbf{1}_{Y>\theta}Y\right) - m}{v}\,\middle|\, Y\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
p &= \int_{-\infty}^{\theta} F_{NIG(c)}\left(\frac{K-\sqrt{a}Y-m}{v}\right) f_{NIG(1)}(Y)dY \\
&\quad + \int_{-\infty}^{\theta} F_{NIG(c)}\left(\frac{K-\sqrt{b}Y-m}{v}\right) f_{NIG(1)}(Y)dY.
\end{aligned}
\tag{6.17}
$$

In order to solve this Equation the constant c has to be specified. c has to be chosen such that in the case where $a = b$ we are back to the double NIG model without factor loadings. In this case the distribution of ε_n had been specified as:

$$
\varepsilon_n \sim NIG\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha\right)
$$

Therefore we can choose the constant c to be either of the following

$$
c = \frac{\sqrt{1-a}}{\sqrt{a}} \qquad c = \frac{\sqrt{1-b}}{\sqrt{b}} \qquad c = \frac{\sqrt{1-\frac{\mathbb{V}(a(Y)Y)}{\gamma}}}{\sqrt{\frac{\mathbb{V}(a(Y)Y)}{\gamma}}}
$$

Then, having specified c, the default trigger levels K can be found numerically from Equation 6.17.

6.4.2 Large portfolio approximation in the double NIG one factor copula model with random factor loadings

As in the Gaussian case the individual conditional default probability is obtained by:

$$
\begin{aligned}
p(y) &= \mathbb{P}(X_1 \le K \mid Y) \\
&= \mathbb{P}\left(\varepsilon_1 \le \frac{K-a(Y)Y-m}{v}\right) \\
&= F_{NIG(c)}\left(\frac{K-1_{Y\le\theta}\sqrt{a}Y-1_{Y>\theta}\sqrt{b}Y-m}{v}\right).
\end{aligned}
\tag{6.18}
$$

Then as shown in Section 6.3.2 the large portfolio approximation justifies that conditional on $Y = y$ the loss X, the fraction of defaulted securities in the portfolio, is given by $p\,(y)$.

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(X \geq x) &= \mathbb{P}\left(p\,(y) \geq x\right) \\
&= \mathbb{P}\left(\frac{K - a(Y)Y - m}{v} \geq F^{-1}_{NIG(c)}(x)\right) \\
&= \mathbb{P}\left(a(Y)Y \leq K - v\, F^{-1}_{NIG(c)}(x) - m\right)
\end{aligned} \tag{6.19}$$

denote by

$$\Omega(x) \;:=\; K - v\, F^{-1}_{NIG(c)}(x) - m$$

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}(X \geq x) &= \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(a(Y)Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= \mathbb{P}\left(\sqrt{a}Y \leq \Omega(x)\,,\, Y \leq \theta\right) + \mathbb{P}\left(\sqrt{b}Y \leq \Omega(x)\,,\, Y > \theta\right) \\
&= F_{NIG(1)}\left(\min\left(\frac{\Omega(x)}{\sqrt{a}}\,,\, \theta\right)\right) \\
&\qquad + 1_{\frac{\Omega(x)}{\sqrt{b}} > \theta}\left(F_{NIG(1)}\left(\frac{\Omega(x)}{\sqrt{b}}\right) - F_{NIG(1)}(\theta)\right).
\end{aligned}$$

and hence the cumulative loss distribution

$$\begin{aligned}
F_\infty(x) &= 1 - \lim_{N\to\infty} \mathbb{P}(X \geq x) \\
&= 1 - \left[F_{NIG(1)}\left(\min\left(\frac{\Omega(x)}{\sqrt{a}}\,,\, \theta\right)\right)\right. \\
&\qquad \left. + \, 1_{\frac{\Omega(x)}{\sqrt{b}} > \theta}\left(F_{NIG(1)}\left(\frac{\Omega(x)}{\sqrt{b}}\right) - F_{NIG(1)}(\theta)\right)\right].
\end{aligned}$$

Chapter 7

Numerical results: pricing the DJ iTraxx

In order to compare the properties of the four large portfolio models introduced in Chapter 6 an example of tranched DJ iTraxx Europe with 5 years to maturity is considered.

The empirical testing of the four models is done with the data set used in the work of Kalemanova et al. [15]. The advantage of using the same data set was that it allowed to check that the double NIG and the Gauss one factor model were implemented correctly. Since such an implementation is not trivial any more and involved hundreds of lines of Matlab Code a backtesting of this sort is essential.

The reference portfolio consists of 125 credit default swap names with equal weight. The tranches have attachment/detachment points at 3%, 6%, 9%, 12% and 22%. The investors of the tranches receive quarterly spread payments on the outstanding notional and compensate for losses when they hit the tranche they are invested in. The investor of the equity tranche receives an upfront fee that is quoted in the market and an annual spread of 500bp quarterly. The settlement date of the third series of this index is 20. September 2005 and matures on 20. September 2010. The market quotes of iTraxx tranches at 11.April 2005 are considered here. The average CDS spread of the corresponding CDS portfolio with 5 years maturity is 37.5bp at this day. The constant recovery is assumed to be 40%. (For more details on the composition and structure of DJ iTraxx Europe see www.iboxx.com).

The relative difference of our numerical implementation compared to the one in Kalemanova et al. [15] was for all tranches of the Gauss and double NIG one factor copula model less than 1.5 bp.

The parameters α and β for the double NIG model were set as estimated by Kalemanova et al. [15].

The parameters a, b and θ of Gauss model with stochastic factor loadings were optimized 1) such that squared deviation of model value minus observed market value was minimized over all tranches, and 2) such that the squared deviation of the model value minus double NIG model value was minimized over all tranches. This allowed to answer two questions: 1) whether introducing stochastic factor loadings improved the fit of the Gauss model and 2) how the fit of the Gauss stochastic factor model differed from the double NIG model.

In the double NIG model with stochastic factor loadings the parameters a, b and θ were optimized to produce the smallest absolute and relative error from the market data.

7.1 iTraxx tranche prices

The market quotes of the iTraxx tranches as well as the prices generated by the four models are summarized in Table 7.1. The price for the equity tranche is the same for all models, since this tranche was used for calibration to the market quotes. The values in the row called 'absolute error' in Table 7.1 is the sum of all absolute differences of model generated tranche prices and market quotes for each model. The 'relative errors' are percentage differences of model generated tranche prices and market quotes summed over all tranches. The 'Stoch Gauss 1' is the Gauss model with stochastic factor loadings where the relative error, hence the percentage difference in tranche prices was minimized. In the 'Stoch Gauss 2' model the difference to the double NIG model tranche prices was minimized.

Numerical results: Gauss and double NIG model

The Gauss one factor copula overprices tranches 3-6% to 9-12% and underprices the most senior tranche. The double NIG one factor copula model prices tranches 3-6% to 9-12% very accurately, but underestimates the most senior tranche.

	Market	Gauss	Stoch Gauss 1	Stoch Gauss 2	Double NIG	Stoch NIG
0 – 3%	24.7 %	24.7 %	24.7 %	24.7 %	24.7 %	24.7 %
3 – 6%	160 bp	241 bp	196 bp	177 bp	160 bp	159 bp
6 – 9%	49 bp	73 bp	50 bp	42 bp	52 bp	51 bp
9 – 12%	22,5 bp	25 bp	19 bp	21 bp	24 bp	22 bp
12 – 22%	13,75 bp	4 bp	15 bp	19 bp	8 bp	11 bp
absolute error		117 bp	42 bp	31 bp	10 bp	6 bp
relative error		101,1 %	8,4 %	18.2 %	18,3 %	4.2 %
correlation		19.47 %			25,59 %	
α					1.2558	1.2558
β					-0.2231	-0.2231
a			0.485	0.524		0.3
b			0.15645	0.14		0.2
θ			-2.5	-2.4		-2.4

Table 7.1: CDO tranche prices generated by the four models

Numerical Results: Gauss model with stochastic factor loadings

The introduction of stochastic factor loadings in the Gauss one factor copula model improved the model fit to market values dramatically. The market values are closest to model prices with a low, negative threshold θ and a large difference in factor loadings. The price for the most senior tranche is higher in the stochastic correlation model because a large part of the value comes from the low probability disaster state with very high factor loadings. It is thus possible that the observed, relatively high premiums of senior tranches are due to a market perception of the existence of a disaster state in which effective correlations are high. This perception does not necessarily conflict with the correlation values computed from historical data. If the sampling period spans several business cycles, the resulting correlations will be weighted averages of correlations in bear and bull markets.

Even though the stochastic factor loadings model provided quite a good fit to market data, the first four tranches could not be fitted as accurately as with the double NIG model. However, in terms of relative error the stochastic factor loadings model produced interesting results. Also note that both, the double NIG and the Gauss model with stochastic factor loadings use the same number of parameters (3). If one is correlation trading on junior tranches as in the example explained in Section 4.2.1 the double NIG model should be preferred as it estimates the first four tranches very well.

Numerical Results: Double NIG model with stochastic factor loadings

The double NIG model with stochastic factor loadings produced a very good fit to market data, improving the fit of the double NIG model. However, such a good fit is at the expense of a slower calculation speed (due to non-stability under convolution in this model extension) and more parameters.

7.2 Loss distributions of the four models

All modified models redistribute risk out of the lower end of the loss distribution to its higher end (see Fig. 7.1). The double NIG model and the stochastic factor loadings model (in both the Gauss and the NIG case) differ from the distribution of the Gauss model in the way required to generate a skew: lower probability of zero loss and a fatter upper tail.

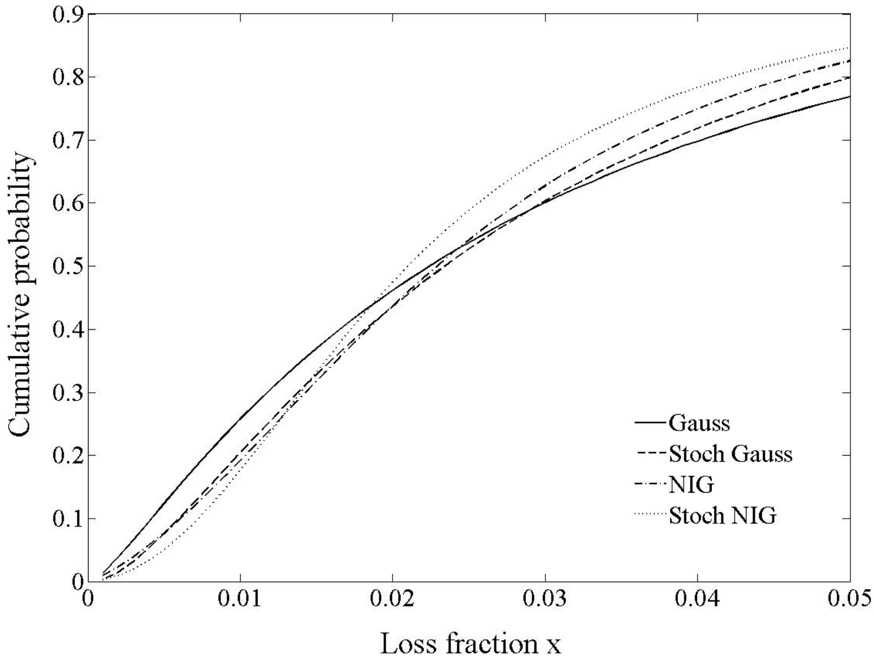

Figure 7.1: Plot of the left tail of the loss distributions produced by the four models with the parameters given in Table 7.1.

7.3 Final remarks and conclusion

In this Chapter the four models developed in Chapter 6 were tested trying to reprice tranches of DJ iTraxx Europe. As expected the model generally used by practitioners, the Gauss one factor copula model, performed badly. Introducing random factor loadings in a very simple way, namely by a two point distribution, proofed to be quite successful. The overpricing of the 3-6% tranche was reduced and the price of the most senior tranche was raised. However, it was not possible to get an as good result for the first four tranches as with the double NIG model. Even though, in the Gauss model with stochastic factor loadings one looses the nice property of stability under convolution for the critical variables X_n, the determination of the default thresholds K is still sufficiently fast.

The double NIG model provides convincing results, since it is able to price the first four tranches extremely accurately. It is a bit slower than the Gauss model with stochastic factor loadings, due to the complexity of the NIG density. However, using the right numerical techniques it should not be too difficult to make it fast enough to meet practitioners need for speed. The great advantage, that makes this model much faster than other more complex models (such as the double t model), is the stability under convolution of the NIG distribution.

The double NIG model with stochastic factor loadings produced very convincing results, even though at the expense of numerical efficiency.

It should by now have become clear that the Gauss one factor model is not the right model to price CDOs and that we are in desperate need for a convincing alternative. In this thesis, three promising models for pricing synthetic CDOs were for the first time compared to each other and to the Gauss one factor model. Further tests using current market data are necessary before we can conclude on a superiority of any of the three models.

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[1]This paper represents the views of the author and should not be interpreted as reflecting the views of my employer or other members of its staff.

Appendix A

Portfolio loss distribution assuming a normal inverse Gaussian copula

If the vector of the critical variables (asset returns) is of the form:

$$\underline{X} = \underline{m}(W) + \sqrt{W}\underline{Z}$$

where

$$\underline{Z} = BY + \underline{\epsilon}$$

and

$$Y \sim N(0, \sigma)$$

$$W \sim IG$$

$$\epsilon \sim N(0, 1)$$

then

$$\underline{X} \sim mNIG$$

$\underline{X}$ is multivariate normal inverse Gaussian distributed, i.e. has normal inverse Gaussian copula.

The portfolio loss distribution is then

$$P\left[X \leq \Theta\right] = \lim_{N \to \infty} \sum_{n=0}^{[N\Theta]} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\begin{array}{c} N \\ n \end{array} \right) \left[\Phi\left(\epsilon\right)\right]^{n} \left[1 - \Phi\left(\epsilon\right)\right]^{N-n} d\Phi(y)\, dG(w) \tag{A.1}$$

with

$$\epsilon = \frac{\left(K - \mu - w\gamma\right) w^{-\frac{1}{2}} - \sqrt{\beta} y}{\sqrt{1-\beta}} \tag{A.2}$$

where Φ denotes the standard normal distribution and G the inverse Gaussian distribution.

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