Valuation of Arithmetic Average of Fed Funds Rates and Construction of the US dollar Swap Yield Curve

Katsumi Takada

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Abstract

Arithmetic averages of Fed Funds (FF) rates are paid on the FF leg of a FF-LIBOR basis swap, while the FF rates are paid with daily compounding in an overnight index swap. We consider here how to value the arithmetic average of FF rates and calculate convexity adjustment terms relative to daily compounded FF rates. FF-LIBOR basis swaps are now the critical calibration instruments for traders to construct the US dollar swap yield curve. We also show how it is constructed in practice.

1 Introduction

An interest rate swap (IRS), an interest rate basis swap (IRBS) and a cross currency basis swap (CCBS) are actively traded in the dealers' swap market. An IRS is the most fundamental interest rate product where fixed rates are exchanged for LIBOR rates. With an IRBS 2 different floating rates in the same currency are exchanged. Those 2 rates can be LIBORs with different tenors or different kinds of floating rates, for example, 3-month LIBOR vs 6-month LIBOR or the overnight (ON) rate vs LIBOR. A CCBS exchanges 3-month LIBORs in non-US dollar for US dollar 3-month LIBORs with the initial and final notional exchanges[1].

Since those swaps traded in the dealers' market are now fully collateralized backed up by the credit support annex (CSA) to an ISDA master agreement or through the settlements at LCH.clearnet, it has become common practice for traders to construct an OIS discounting curve and multiple forward curves for each LIBOR tenor (for example, see Bianchetti (2010) or Pallavicini and Tarenghi (2010)).

In the dealers' market ON rates are traded against LIBOR rates more actively in the form of an IRBS rather than against fixed rates in the form of an overnight indexed swap (OIS). Contrary to the popular belief, the OIS discounting curve is not constructed from quoted OIS rates. Rather

[1] US-dollar notional is reset at the FX spot rate at the start date of each interest period to mitigate counterparty risk.

so that $\delta(T_s, T_e) = \sum_{k=1}^{K} \delta_k$. Hence the interest amounts over $[T_s, T_e]$ are calculated as

$$N * R_c(T_s, T_e) * \delta(T_s, T_e) = N * \left[\prod_{k=1}^{K} (1 + \delta_k C_k) - 1\right]$$

and

$$N * R_a(T_s, T_e) * \delta(T_s, T_e) = N * \sum_{k=1}^{K} \delta_k C_k$$

where N is a notional amount.

Interest rates are subject to the rate convention. The relationship between a simple compounded interest rate R and a continuously compounded interest rate r with the same day count convention is given by

$$1 + \delta R = e^{\delta r} \tag{1}$$

where δ is the day count fraction for the interest period. If we take R to be the ON rate, then the compounded amount over the interest period $[T_s, T_e]$ is calculated as

$$\prod_{k=1}^{K} (1 + \delta_k C_k) = e^{\sum_{k=1}^{K} \delta_k c_k} \tag{2}$$

where c_k is a continuously compounded ON rate which satisfies $1 + \delta C_k = e^{\delta c_k}$. Note that there is a limiting relationship which says:

$$\lim_{\delta \searrow 0} (1 + \delta R)^{1/\delta} = e^{R}$$

This means that for sufficiently small δ, the simple compounded interest rate reasonably approximates the continuously compounded rate. Since the ON rate has a small δ, (2) can be approximated by

$$\prod_{k=1}^{K} (1 + \delta_k C_k) = e^{\sum_{k=1}^{K} \delta_k C_k}$$

Hence AAON can be described as:

$$\begin{aligned} R_a(T_s, T_e) &= \frac{1}{\delta(T_s, T_e)} \log \prod_{k=1}^{K} (1 + \delta_k C_k) \\ &= \frac{1}{\delta(T_s, T_e)} \log (1 + \delta(T_s, T_e) R_c(T_s, T_e)) \end{aligned} \tag{3}$$

To see how precise this approximation is, we calculate the RHS and LHS of (3) with $\delta_k = \frac{1}{360}$ when $K = 360$ and $\delta(T_s, T_e) = 1$ in Table 1 and when $K = 90$ and $\delta(T_s, T_e) = 0.25$ in Table 2. Table 1 describes the results when the length of an interest period is 1 year and Table 2 does when the length is 3 month. In each table, we show the results when the sequence of ON rates starts with 10% and 1%. For each starting value C_1, we generated ON rates over the interest

3 Valuation of variable rates on the ON-rate leg

Fujii et al (2009) and Piterbarg (2010) have shown that since the interest rate accruing on the collateral account is the ON rate, the time t value of the collateralized European derivative $V(t)$ whose payoff at the maturity T is X can be written as

$$V(t) = E_t^Q \left[e^{-\int_t^T c(u)du} X \right] \tag{5}$$

for $t \leq T$, where $E_t^Q[\cdot]$ denotes the time t conditional expectation under the risk-neutral measure Q. Note the short rate form of the ON rate c replaces the risk-free short rate r in the usual pricing formula. This is good news because we do not observe the risk-free rate r in practice. The notion of "OIS discounting" comes from (5). We denote the time t price of the collateralized discount bond with the maturity T by

$$D(t;T) = E_t^Q \left[e^{-\int_t^T c(u)du} \right] \tag{6}$$

for $t \leq T$.

Denoting further the initial discount factor maturing at T by $D(T) \equiv D(0;T)$, the present value of a collateralized DCON over the interest period $[T_s, T_e]$ is given by[3],

$$\begin{aligned} V_c &= E^Q \left[e^{-\int_0^{T_e} c(u)du} \delta(T_s,T_e) R_c(T_s,T_e) \right] \\ &= E^Q \left[e^{-\int_0^{T_e} c(u)du} \left[\prod_{k=1}^{K} (1 + \delta_k C_k) - 1 \right] \right] \\ &= D(T_s) - D(T_e) \\ &= \delta(T_s,T_e) O(0;T_s,T_e) D(T_e) \end{aligned} \tag{7}$$

where

$$O(t;T_s,T_e) = \frac{1}{\delta(T_s,T_e)} \left[\frac{D(t;T_s)}{D(t;T_e)} - 1 \right], \; t \leq T_s \tag{8}$$

is a time t forward rate for the DCON and $E^Q[\cdot]$ denotes the initial unconditional expectation under the risk-neutral measure Q. The relation between c and C_k, $E_{t_k}^Q \left[e^{-\int_{t_k}^{t_{k+1}} c(u)du} \right] = \frac{1}{1+\delta_k C_k}$ is used from the 2nd to 3rd equation in (7). Note that (8) is exactly the same as the textbook forward LIBOR formula. Although the DCON is not completely fixed until the end of the interest period, for valuation purposes it can be treated as fixed at the OIS rate for the length of $[T_s, T_e]$ which is observed at the start date T_s. In this sense, $O(t;T_s,T_e)$ is called a time t forward OIS rate maturing at T_s as the spot OIS rate over $[T_s, T_e]$, in the same way that the forward LIBOR matures at T_s as the spot LIBOR[4].

[3] We used here the risk neutral expectation operator, but in the case of the DCON, the static replication argument also applies.

[4] More precisely, forward OIS matures on the trading date of the OIS while forward LIBOR matures on the

This is consistent with (3). Note also that we do not in fact need the final approximation of (11) to get (13). We plot $\log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$ against $\delta(T_s, T_e)O(0; T_s, T_e)$ in Figure 1. It is clear from Figure 1 that there is convexity in the AAON relative to the DCON. Also we have

$$\delta(T_s, T_e)O(0; T_s, T_e) \geq \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)), \tag{14}$$

with equality when $O = 0$.

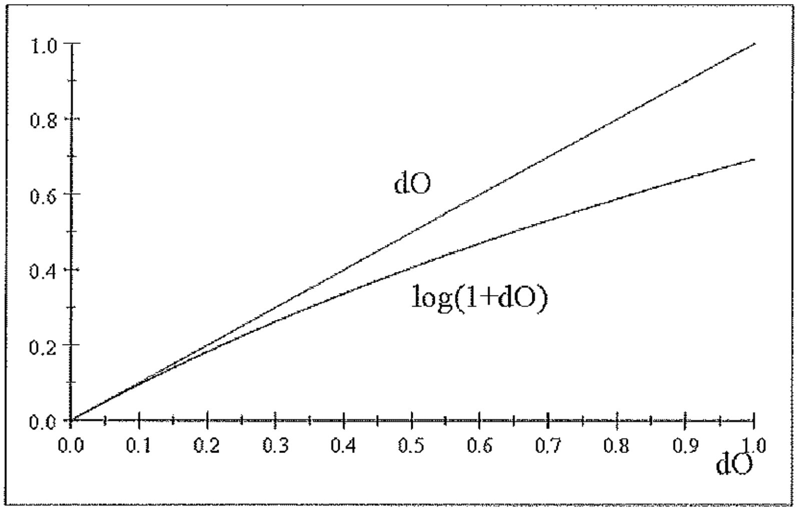


Figure 1: $\log(1 + \delta O)$ is plotted against δO.

Consider a strategy where an AAON is paid against its forward rate (12), and, as a delta hedge, the DCON is received against its forward rate $O(0; T_s, T_e)$. If the forward rate for the AAON is $\frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$ as in (13), the net payoff curve paid at T_e against the forward OIS rate is convex below and the minimum point is zero at the initial forward OIS rate. When the forward OIS rate moves in any direction, this strategy makes money. To avoid arbitrage,

$$O_a(0; T_s, T_e) < \frac{1}{\delta(T_s, T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) \tag{15}$$

should hold. Combining (14) with (15), we have:

$$O_a(0; T_s, T_e) < \frac{1}{\delta(T_s, T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) \leq O(0; T_s, T_e) \tag{16}$$

We conclude that the forward rate of the AAON over $[T_s, T_e]$ is smaller than that of the DCON over the same interest period by 2 convexity correction terms. The first convexity correction is static in the sense that it exists even when the forward OIS rate is not volatile at all, and its value is $O(0; T_s, T_e) - \frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e))$. This corresponds to the convexity correction in (4). The other convexity term is dynamic in the sense that it occurs due to the volatility of the forward OIS rate, and its value is $\frac{1}{\delta(T_s,T_e)} \log(1 + \delta(T_s, T_e)O(0; T_s, T_e)) - O_a(0; T_s, T_e)$. We next

5 Pricing of arithmetic average of ON rates with Hull White model

We decompose the time T_e-forward value of the AAON, $E^{T_e}\left[\int_{T_s}^{T_e} c(u)du\right]$ in (10) into 2 parts and calculate them with the single-factor HW model.

$$E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} c(u)du\right] = E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] + E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} c(T_s;u)du\right] \tag{21}$$

Using (18) and (19), the 1st term of the RHS in (21) becomes:

$$E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] = \int_{T_s}^{T_e} \left(H(u,u) - H(T_s,u)\right)du + E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} \left(Z(u) - e^{-\int_{T_s}^{u}\lambda(x)dx}Z(T_s)\right)du\right] \tag{22}$$

Since

$$\begin{aligned} Z(u) &= e^{-\int_{T_s}^{u}\lambda(x)dx}Z(T_s) + \int_{T_s}^{u}\sigma(s,u)dW^Q(s) \\ &= e^{-\int_{T_s}^{u}\lambda(x)dx}Z(T_s) + \int_{T_s}^{u}\sigma(s,u)\left(dW^{T_e}(s) - \nu(s,T_e)ds\right) \end{aligned}$$

(22) further becomes:

$$E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} \left(c(u) - c(T_s;u)\right)du\right] = -\int_{T_s}^{T_e}\int_{T_s}^{u}\sigma(s,u)\left(\nu(s,T_e) - \nu(s,u)\right)dsdu \tag{23}$$

The 2nd term of the RHS in (21) can be written with the HW model as:

$$E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} c(T_s;u)du\right] = \int_{T_s}^{T_e} c(0;u)du + \int_{T_s}^{T_e} H(T_s;u)du + E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} e^{-\int_{T_s}^{u}\lambda(x)dx}Z(T_s)du\right] \tag{24}$$

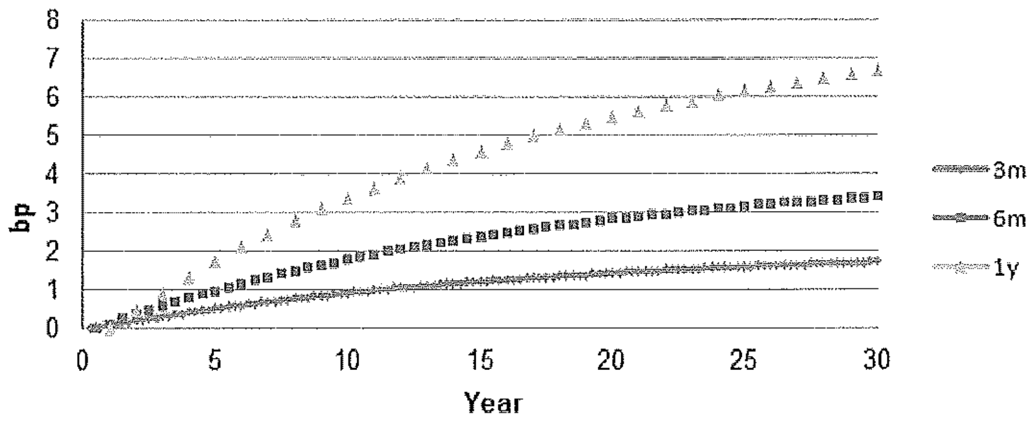
Using

$$\begin{aligned} Z(T_s) &= \int_0^{T_s}\sigma(s,T_s)dW^Q(s) \\ &= \int_0^{T_s}\sigma(s,T_s)\left(dW^{T_e}(s) - \nu(s,T_e)ds\right), \end{aligned}$$

the 3rd term of the RHS in (24) can be explicitly calculated as:

$$E^{\mathrm{T}_e}\left[\int_{T_s}^{T_e} e^{-\int_{T_s}^{u}\lambda(x)dx}Z(T_s)du\right] = -\int_{T_s}^{T_e}\int_0^{T_s}\sigma(s,u)\nu(s,T_e)dsdu$$

Figure 2 : Convexity Adjustment 1, (29) divided by the length of interest period is plotted against each payment date. HW model parameters, (λ,σ)=(0.03,0.01) are used.



interest period	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1.0$
conv adj 2 / δ	0.01bp	0.04bp	0.16bp

Table 3: Convexity adjustment 2, (30) divided by the length of the interest period with $(\lambda, \sigma) = (0.03, 0.01)$.

Table 3 indicates that convexity adjustment 2 is small. Indeed, it is safe to be ignored when the length of the interest period is as small as 3 months or 6 months.

Figure 2 plots (29) divided by the interest period length (1, 0.5 and 0.25) for each payment date T_e. The reason that convexity adjustment 1 is larger for longer interest periods is that the convexity of $\delta^{-1}\log(1+\delta O)$ is larger with respect to the OIS rate O. With the same convexity level, the larger the time to maturity of the forward OIS rate is, the more benefit one can get from the convexity.

We next plot in Figure 3 convexity adjustment 1 on the basis of the discount-factor weighted average up to T_M (x-axis in Figure 3) to get the same scale as a quoted basis swap spread. The numbers are calculated for each interest period $\delta = 1, 0.5$ and 0.25 as

$$\frac{\sum_{i=1}^{M} \mathrm{ConvAdj}(T_i)/\delta * D(T_i)}{\sum_{i=1}^{M} D(T_i)} \tag{31}$$

where $T_1 = \delta$, $T_i - T_{i-1} = \delta$, and $\mathrm{ConvAdj}(T_i)$ is an adjustment term (29) paid at T_i

Figure 3 shows that the calibrating instruments of the basis swaps where FF arithmetic average is exchanged for 3-month LIBOR have small but distinct convexity adjustment, say, the 10-year swap has about 0.5bp of convexity adjustment and the 30-year swap has about 1bp in terms of quoted basis swap spread. Considering the bid-offer spread of basis swaps of FF and 3-month where $O(t;t,T)$ is the spot OIS rate over the interest period $[t,T]$, and is given by

$$O(t;t,T)=\frac{1}{\delta(t,T)}\left(\frac{1}{D(t;T)}-1\right) \tag{32}$$

Any twice differentiable payoff $f(O)$ can be re-written as (see, for example, Carr and Madan (2002))

$$f(O)=f(\kappa)+f'(\kappa)(O-\kappa)+\int_0^{\kappa} f''(K)(K-O)^{+}dK+\int_{\kappa}^{\infty} f''(K)(O-K)^{+}dK \tag{33}$$

Putting $f(O)=\log(\delta O+1)$ and applying (33) to it, we have

$$\log(\delta O+1)=\log(\delta\kappa+1)+\frac{\delta}{\delta\kappa+1}(O-\kappa)-\int_0^{\kappa}\frac{\delta^2}{(\delta K+1)^2}(K-O)^{+}dK-\int_{\kappa}^{\infty}\frac{\delta^2}{(\delta K+1)^2}(O-K)^{+}dK \tag{34}$$

Applying T-forward measure expectation to the both side of (34) and taking $\kappa=O(0;t,T)$, we have the 2nd term of the RHS in (21) as

$$E^{\mathrm{T}}\left[\int_t^T c(t;u)du\right]=\log(\delta O(0;t,T)+1)-\int_0^{O(0;t,T)}\frac{\delta F_o(K)}{(\delta K+1)^2}dK-\int_{O(0;t,T)}^{\infty}\frac{\delta C_o(K)}{(\delta K+1)^2}dK \tag{35}$$

where $E^T[O(t;t,T)]=O(0;t,T)$ is used, $F_o(K)$ and $C_o(K)$ denote respectively time T-forward value of OIS foorlet and caplet with strike K and the interest period $[t,T]$. Comparing (35) with (25), it follows that the sum of the 2nd and 3rd terms of the RHS of (35) corresponds to convexity adjustment 1 in the previous section.

Since the market is not so matured that OIS caps/floors are traded, we will replace in (35) OIS caplets/floorlets with LIBOR caplets/floorlets which are traded actively in the market.

The time T-forward value of OIS caplet with strike K is given by:

$$\begin{aligned}\delta(t,T)E^{\mathrm{T}}\left[(O(t;t,T)-K)^{+}\right]&=E^{\mathrm{T}}\left[\left(\frac{1}{D(t;T)}-1-\delta(t,T)K\right)^{+}\right]\\&=E^{\mathrm{T}}\left[(D_{l-d}(t;T)(1+\delta(t,T)L(t;t,T))-1-\delta(t,T)K)^{+}\right]\end{aligned} \tag{36}$$

where we used the spot OIS formula (32) and spot LIBOR formula:

$$\begin{aligned}L(t;t,T)&=\frac{1}{\delta(t,T)}\left(\frac{1}{D_l(t;T)}-1\right)\\&=\frac{1}{\delta(t,T)}\left(\frac{1}{D_{l-d}(t;T)D(t;T)}-1\right)\end{aligned}$$

where $D_l(\cdot;T)$ is the T-maturity LIBOR discount factor, $D_{l-d}(\cdot;T)$ is the LIBOR-discount rate initial time be time 0 and the swap spot date be T_0. US dollar discount factor maturing at T is denoted by:

$$D(T) \equiv D(0;T) = E^Q\left[e^{-\int_0^T c(u)du}\right]$$

where c is the short rate form of the Fed Funds rate.

We assume here for simplicity that IRSs and IRBSs are the only market instruments for constructing the yield curve[8]. Let S_N denote an annual-money IRS rate for the tenor of N year. Since the value of the IRS at the quoted IRS rate is zero, the IRS rate satisfies

$$S_N \sum_{i=1}^{N} \delta(T_{4(i-1)}, T_{4i}; act/360) D(T_{4i}) = \sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) L(0; T_{i-1}, T_i) D(T_i) \tag{39}$$

for each $N = 1, 2, \cdots$, where the collateralized 3-month forward LIBOR is given by

$$L(0; T_{i-1}, T_i) = \frac{1}{\delta(T_{i-1}, T_i; act/360)} \left(\frac{D_{3l-d}(T_{i-1}) D(T_{i-1})}{D_{3l-d}(T_i) D(T_i)} - 1 \right) \tag{40}$$

Here $D_{3l-d}(T)$ is the T-maturity spread discount factor of 3-month LIBOR minus the discount rate.

Let B_N denote a basis swap spread added to the averaged FF rates against 3-month LIBOR in the IRBS. The basis swap spread satisfies

$$\sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) L(0; T_{i-1}, T_i) D(T_i) = \sum_{i=1}^{4N} \delta(T_{i-1}, T_i; act/360) \left\{ O_a(0; T_{i-1}, T_i) + B_N \right\} D(T_i), \tag{41}$$

for each $N = 0.5, 0.75, 1, 2, \cdots$, where the collateralized 3-month forward "arithmetically averaged" OIS rate is given by

$$O_a(0; T_{i-1}, T_i) = \frac{1}{\delta(T_s, T_e; act/360)} \left(\log \frac{D(T_{i-1})}{D(T_i)} - \text{Convexity Adj}_i \right) \tag{42}$$

Here the convexity adjustment term can be calculated through some model as (28) or with cap/floor market prices as (38). Although theoretically the yield curve and the volatility curve are simultaneously determined through the convexity adjustment of the AAON, this term is calculated beforehand and is an constant input to the yield curve construction in practice. Convexity Adj $= 0$ may be justified because the interest period on the FF leg is 3 months and its convexity correction may be small in the current market as we saw in the HW model example.

From (39), (41) and with the proper interpolating method, we can solve for the discounting curve $\{D(t)\}_{t>0}$ and spread curve of 3-month LIBOR minus discount rate, $\{D_{3l-d}(t)\}_{t>0}$ subject to $D(0) = 1$ and $D_{3l-d}(0) = 1$. Note that $\{D(t)\}_{t>0}$ is a so-called OIS discounting curve and

[8] We ignore MM, LIBOR futures, and FRAs.

References

[1] Bianchetti M, 2010, Two curves, *one prices: pricing & hedging interest rate derivatives decoupling forwarding and discounting yield curves*, SSRN eLibrary

[2] Carr P and Madan D, 2002, *Towards a Theory of Volatility Trading*, Volatility, Risk Publications, pages 417-427

[3] Credit Suisse Fixed Income Research, 2010, *A guide to the Front-End and Basis Swap Markets*

[4] Fujii M, Shimada Y and Takahashi A, 2009, *A note on construction of multiple swap curves with and without Collateral*, SSRN eLibrary

[5] Pallavicini A and Tarenhi M, 2010, *Interest-rate modeling with multiple curves*, SSRN eLibrary

[6] Piterbarg V, 2010, *Funding beyond discounting: collateral agreements and derivatives pricing*, Risk February, pages 97-102