

# A Survey and Some Generalizations of Bessel Processes

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## Abstract

Bessel processes play an important role in financial mathematics because of their strong relation to financial processes like geometric Brownian motion or CIR processes. We are interested in the first time Bessel processes and more generally, radial Ornstein–Uhlenbeck processes hit a given barrier. We give explicit expressions of the Laplace transforms of first hitting times by (squared) radial Ornstein–Uhlenbeck processes, i. e., CIR processes. As a natural extension we study squared Bessel processes and squared Ornstein–Uhlenbeck processes with negative dimensions or negative starting points and derive their properties.

Keywords: First hitting times; CIR processes; Bessel processes; radial Ornstein–Uhlenbeck processes; Bessel processes with negative dimensions

## 1 Introduction

Bessel processes have come to play a distinguished role in financial mathematics for at least two reasons, which have a lot to do with the models being usually considered. One of these models is the Cox–Ingersoll–Ross (CIR) family of diffusions, also known as square-root diffusions, which solve

$$dX_t = (a + bX_t) dt + c \sqrt{|X_t|} dB_t, \quad (1)$$

with  $X_0 = x_0 \geq 0$ ,  $a \geq 0$ ,  $b \in \mathbb{R}$ ,  $c > 0$  and  $(B_t)$  standard Brownian motion. For every given value  $x_0 \geq 0$ , equation (1) admits a unique solution; this solution is strong, i.e. adapted with respect to the natural filtration of  $(B_t)$ , and takes

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values in  $[0, \infty)$ . If  $a = 0$  and  $x_0 = 0$ , the solution of (1) is  $X_t \equiv 0$ , and from the comparison theorem for one-dimensional diffusion processes, see Revuz–Yor [49] Theorem IX.(3.7), we deduce  $X_t \geq 0$  for  $a \geq 0$ ,  $x_0 \geq 0$ . Hence, in this case the absolute value in (1) can be omitted. Cox, Ingersoll and Ross [10] proposed this family of diffusions for modelling short term interest rates. In recent financial literature they are often studied from different points of view, or serve as important reference processes, see for instance Chen–Scott [8], Deelstra–Parker [12], Delbaen [13], Duffie–Singleton [17], Frydman [22] and Leblanc [36, 37]; they are used for modeling stochastic volatility, see e. g. Ball [3], Genotte and Marsh [24] and Heston [28]. The other even more fundamental model is geometric Brownian motion, standardly used as a model for stock prices

$$S_t = S_0 \exp(\nu t + \sigma B_t), \quad (2)$$

with  $\nu, \sigma \in \mathbb{R}$ ,  $(B_t)$  standard Brownian motion. In both cases,  $X$  and  $S$  can be represented in terms of (squares of) Bessel processes. We recall the definition of (squares of) Bessel processes, see Revuz–Yor [49], Chapter XI.

**Definition 1** For every  $\delta \geq 0$  and  $x_0 \geq 0$  the unique strong solution to the equation

$$X_t = x_0 + \delta t + 2 \int_0^t \sqrt{|X_s|} dB_s \quad (3)$$

is called the square of a  $\delta$ -dimensional Bessel process started at  $x_0$  and is denoted by  $\text{BESQ}_{x_0}^\delta$ .

As in the case of CIR processes (1) with  $a \geq 0$  and  $x_0 \geq 0$ , using the comparison theorem, we deduce  $X_t \geq 0$  in (3) from the fact that  $X_t$  is identically zero if  $\delta = 0$ ,  $x_0 = 0$ . Thus, the absolute value under the square root may a posteriori be omitted in (3). We call the number  $\delta$  the *dimension* of  $\text{BESQ}^\delta$ . This notation arises from the fact that a  $\text{BESQ}^\delta$  process  $X_t$  can be represented by the square of the Euclidean norm of  $\delta$ -dimensional Brownian motion  $B_t$ :  $X_t = |B_t|^2$ . The number  $\nu \equiv \delta/2 - 1$  is called the *index* of the process  $\text{BESQ}^\delta$ . For  $\delta \geq 2$ ,  $\text{BESQ}^\delta$  processes will never reach 0 for  $t > 0$ , and for  $0 \leq \delta < 2$  they reach 0 almost surely.

**Definition 2** The square root of  $\text{BESQ}_{y^2}^\delta$ ,  $\delta \geq 0$ ,  $y \geq 0$  is called the Bessel process of dimension  $\delta$  started at  $y$  and is denoted by  $\text{BES}_y^\delta$ .

For a study of Bessel processes we refer to Revuz–Yor [49] and Pitman–Yor [45, 46], see also Appendix A. A CIR process (1) can be represented as

$$X_t = e^{bt} Y \left( \frac{c^2}{4b} (1 - e^{-bt}) \right), \quad (4)$$

where  $Y$  denotes a squared Bessel process with dimension  $\delta = \frac{4a}{\sigma^2}$ . We remark that this relation is used e. g. by Delbaen and Shirakawa [14] and Szatzschneider [57]. For geometric Brownian motion (2) we have the relation

$$S_t = \rho^{(\frac{\nu}{\sigma^2})} \left( \sigma^2 \int_0^t \exp 2(\nu s + \sigma B_s) ds \right), \quad (5)$$

$t \geq 0$ , where  $\left( \rho^{(\frac{\nu}{\sigma^2})}(u), u \geq 0 \right)$  denotes a Bessel process with index  $\frac{\nu}{\sigma^2}$ , with  $\rho^{(\frac{\nu}{\sigma^2})}(0) = S_0$ . Formula (5) is the so-called Lamperti representation, see Lamperti [34] and also Williams [60]. The Lamperti representation has been very useful e. g. for inference in the Asian option pricing, see Geman and Yor [23] and Yor [63], Chapter 6.

We now make some remarks about the range of the values of the parameters  $a, b, c, \sigma$  and  $\nu$  and some consequences. The signs of  $c$  and  $\sigma$  are irrelevant since  $B \stackrel{(d)}{=} -B$ . We assume  $c, \sigma > 0$ . In fact, by scaling we will restrict ourselves to  $c = 2$  and  $\sigma = 1$ . For  $c = 2$ , CIR processes defined by (1) coincide with squared radial Ornstein–Uhlenbeck processes which are studied in detail in Section 2, see (7), and in the following we will use both notations synonymously. The sign of  $b$  influences the behaviour of a CIR process  $X$ , since in the case  $a > 0$  there exists a unique stationary density of  $X$  only if  $b < 0$ ; we note that stationary CIR processes also have the ergodic property. In the case  $a \geq \frac{c^2}{2} = 2$ , a CIR process starting in  $x_0 \geq 0$  stays strictly positive for  $t > 0$ ; for  $a < 2$ , a CIR process  $X_t$  starting in  $x_0 \geq 0$  hits 0 with probability  $p \in ]0, 1[$  if  $b > 0$  and almost surely if  $b \leq 0$ . Note that in the case  $a > 0$  the boundary 0 is instantaneously reflecting, whereas in the case  $a = 0$ , as soon as a CIR process  $X_t$  hits 0 it is extinct.

Until now, studies of square-root diffusions defined by (1) have always assumed  $a \geq 0$  and  $x_0 \geq 0$ . Under these assumptions we deduce  $X_t \geq 0$  for all  $t \geq 0$  from the comparison theorem for one-dimensional diffusion processes and the absolute value in the square root term can be omitted. But it seems to be a natural extension also to consider  $a < 0$  or to start the process in  $x_0 < 0$ . In such cases one should be already careful about the formulation of the square root term; for the introduction we consider the case  $a < 0$  and  $x_0 > 0$  with the formulation (1). But see also Section 3 for a more general substitution, i. e., we replace  $\sigma(x) \equiv c\sqrt{|x|}$  by  $\tilde{\sigma}(x) \equiv c\sqrt{\alpha x^+ + \beta x^-}$ ,  $\alpha, \beta \geq 0$ . The process  $(X_t)$ ,  $t \geq 0$ , defined by (1) with  $a < 0$ ,  $b \in \mathbb{R}$ ,  $c > 0$  and starting in  $x_0 > 0$  remains in  $\mathbb{R}_+$  until  $T_0 = \inf\{t > 0 | X_t = 0\}$ . Then, since  $Y_t \equiv -X_{T_0+t}$  satisfies

$$dY_t = (-a - bY_t) dt + c\sqrt{|Y_t|} d\tilde{B}_t,$$

with  $(\tilde{B}_t)$  standard Brownian motion, we know that  $Y_t \geq 0$  for all  $t \geq 0$ , thus  $(X_t, t \geq T_0)$  ranges in  $\mathbb{R}_-$  and  $(Y_t)$  is a CIR process (1) with parameters  $-a > 0$ ,  $-b \in \mathbb{R}$  and  $c > 0$ .

Consider equation (5) with  $\nu < 0$ . We know by Dufresne [18] (see also Théorème 1 in Yor [64])

$$\int_0^\infty \exp 2(B_s + \nu s) ds \stackrel{(d)}{=} \frac{1}{2Z_{(-\nu)}},$$

where  $Z_\mu$ ,  $\mu > 0$ , is a Gamma variable with index  $\mu$ , i. e.,

$$P(Z_\mu \in dt) = \frac{t^{\mu-1} e^{-t}}{\Gamma(\mu)} dt.$$

We have  $\rho_u^{(\nu)} \rightarrow 0$  for  $u \rightarrow \int_0^\infty \exp 2(B_s + \nu s) ds$ , and  $\tilde{T}_0 \equiv \inf\{t | \rho_t^{(\nu)} = 0\} = \int_0^\infty \exp 2(B_s + \nu s) ds$ . Thus, as far as one is only interested in  $(S_t, t \geq 0)$  in (2) or (5), there is no need to consider  $(\rho_{\tilde{T}_0+u}, u \geq 0)$ . However, it seems natural to do this, especially when confronted with model (1). The case  $\nu < -1$  corresponds to a Bessel process  $\rho^{(\nu)}$  with dimension  $\delta = 2(\nu + 1) < 0$ . In Section 3, we will derive and discuss properties of negative-dimensional squared Bessel processes, and also squared radial Ornstein–Uhlenbeck processes with  $\delta < 0$ , with starting points in  $\mathbb{R}$ .

The above discussion shows how first hitting times of squared radial Ornstein–Uhlenbeck processes, i.e. CIR processes, may arise. In Section 2, we derive an explicit form of the Laplace transform of first hitting times of (squared) radial Ornstein–Uhlenbeck processes, by exploiting the relation between radial Ornstein–Uhlenbeck processes and Bessel processes. Note that because of the previous discussions of the sign and behaviour of  $X$  in the negative dimensional case we will have no difficulties in computing the Laplace transforms of first hitting times  $T_y$  of a negative dimensional process  $X$  starting in  $x_0$  in all possible cases, say  $x_0 > 0 > y$ .

From a financial point of view we are interested in the quantity

$$E_a[1_{(T_x < t)}(R_t - k)^+],$$

with a radial Ornstein–Uhlenbeck process  $(R_t)$  starting in  $a \geq 0$ , see Section 2, and  $k \in \mathbb{R}^+$ . We remark that the quantity  $E[1_{(T_x < t)}(S_t - k)^+]$ , expressing values of barrier options with underlying stock price process  $(S_t)$  in (2), is investigated in Chesney et al. [9] by considering Laplace transforms with respect to time. The Laplace transform of  $E_a[1_{(T_x < t)}(R_t - k)^+]$  with respect to time is

$$\begin{aligned} \alpha \int_0^\infty e^{-\alpha t} E_a[1_{(T_x < t)}(R_t - k)^+] dt &= E_a[\alpha \int_{T_x}^\infty e^{-\alpha t} (R_t - k)^+ dt] \\ &= E_a[e^{-\alpha T_x} \alpha \int_0^\infty e^{-\alpha u} (R_{T_x+u} - k)^+ du]. \end{aligned}$$

Using the strong Markov property this equals

$$E_a[e^{-\alpha T_x}] \alpha E_x \left[ \int_0^\infty e^{-\alpha u} (R_u - k)^+ du \right], \quad (6)$$

which immediately leads to the computations of the Laplace transforms of first hitting times of radial Ornstein–Uhlenbeck processes, as well as to that of its resolvent; as for the resolvent we refer to Remark 6 in Section 2.

## 2 First hitting times of radial Ornstein–Uhlenbeck processes

As motivated in the introduction we are interested in finding the law of first hitting times of (squared) radial Ornstein–Uhlenbeck processes. For general discussions of first hitting times of diffusions we refer to Arbib [2], Breiman [7], Horowitz [29], Novikov [41, 42, 43], Pitman and Yor [48], Ricciardi and Sato [50], Rogers [51], Salminen [52], Shepp [54], Siegert [56], Truman and Williams [58] and Yor [61].

First we recall the definition of (squared) radial Ornstein–Uhlenbeck processes. We will use another notation than in (1) which is related to the notation in Definition 1. Let  $(W_t)$  be a one-dimensional Brownian motion,  $\lambda \in \mathbb{R}$ ,  $\delta \geq 0$  and  $z \geq 0$ . The solution to the equation

$$Z_t = z + \int_0^t (\delta - 2\lambda Z_s) ds + 2 \int_0^t \sqrt{|Z_s|} dW_s$$

is unique and strong (see Revuz–Yor [49] Chapter IX §3). Since in the case  $\delta = 0, z = 0$ , the solution is  $Z_t \equiv 0$ , we deduce from the comparison theorem  $Z_t \geq 0$  for all  $\delta \geq 0$ , and hence the absolute value can be omitted; the solution of

$$Z_t = z + \int_0^t (\delta - 2\lambda Z_s) ds + 2 \int_0^t \sqrt{Z_s} dW_s \quad (7)$$

is called a *squared  $\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $-\lambda$*  and its law on  $C(\mathbb{R}_+, \mathbb{R})$  is denoted by  ${}^{-\lambda}Q_z^\delta$ . It is a Markov process; hence, the square root of this process is also a Markov process and is called a  *$\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $-\lambda$* . Denote its law on  $C(\mathbb{R}_+, \mathbb{R})$  by  ${}^{-\lambda}P_x^\delta$  where  $x = \sqrt{z}$ . For  $\delta \geq 2$  and  $z > 0$ , it almost surely does not hit zero. For  $\delta > 1$  it is the solution to the equation

$$dR_t = \left( \frac{\delta - 1}{2R_t} - \lambda R_t \right) dt + dW_t, \quad R_0 = x = \sqrt{z}.$$

For  $\delta = 1$  we have with Itô–Tanaka’s formula

$$|R_t| = |x| - \lambda \int_0^t |R_s| ds + \tilde{W}_t + L_t,$$

where  $(R_t)$  is a  ${}^{-\lambda}P_x^1$  process,  $\tilde{W}_t \equiv \int_0^t \text{sgn}(R_s) dW_s$  is standard Brownian motion and  $L_t$  is the local time of Brownian motion. Using Girsanov’s theorem we obtain

the relationship

$${}^{-\lambda}P_x^\delta|_{\mathcal{F}_t} = \exp \left\{ -\lambda \int_0^t R_s dW_s - \frac{\lambda^2}{2} \int_0^t R_s^2 ds \right\} \cdot P_x^\delta|_{\mathcal{F}_t},$$

where  $P_x^\delta$  denotes the law of a BES $^\delta(x)$  process. Hence for  $\delta < 1$  we obtain from (42)

$$R_t = x + \frac{\delta - 1}{2} \text{p.v.} \int_0^t \frac{ds}{R_s} - \lambda \int_0^t R_s ds + \hat{W}_t,$$

where  $(\hat{W}_t)$  is standard Brownian motion under  ${}^{-\lambda}P^\delta$  and p.v. denotes the principal value.

Our aim is to find the law of

$$T_{x \rightarrow y} \equiv \inf\{t \mid R_t = y\}, \quad (8)$$

the first time a radial Ornstein–Uhlenbeck process  $(R_t)$  with parameter  $-\lambda$  starting in  $x \geq 0$  hits the level  $y$ . We distinguish between the cases  $0 \leq y < x$  and  $0 \leq x < y$ . For  $\delta < 2$ , we have  ${}^{-\lambda}P_x^\delta(T_{x \rightarrow 0} < \infty) > 0$ , that is, the process  $(R_t)$  may reach 0; if  $\delta < 2$  and  $\lambda > 0$ , then  ${}^{-\lambda}P_x^\delta(T_{x \rightarrow 0} < \infty) = 1$ , that is,  $(R_t)$  reaches 0 almost surely and hence every  $y$  almost surely,  $0 \leq y < x$ . For  $\delta \geq 2$ , we have  ${}^{-\lambda}P_x^\delta(T_{x \rightarrow y} < \infty) = 1$  almost surely for every  $y \geq x$ .

**Remark 1** Since  $T_{x \rightarrow y} = \inf\{t \mid R_t = y\} = \inf\{t \mid R_t^2 = y^2\}$ , once the law of (8) is obtained we have the law of the first time a squared radial Ornstein–Uhlenbeck process, i. e., a CIR process, starting in  $a (= x^2)$  hits  $b (= y^2)$ .

Call  ${}^{-\lambda}P_x^\delta$  the law of a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $-\lambda$ . The density  ${}^{-\lambda}p_x^\delta(t)$  of the first hitting time of 0 of a radial Ornstein–Uhlenbeck process is calculated in Elworthy–Li–Yor [21] (see Corollary 3.10) by using a time reversal argument from  $T_{x \rightarrow 0}$ , that is, for  $y = 0$  the problem is already solved:

$${}^{-\lambda}p_x^\delta(t) = \frac{x^{2-\delta}}{2^\nu \Gamma(\nu)} \exp \left[ \frac{\lambda}{2} (\delta t + x^2 (1 - \coth(\lambda t))) \right] \left[ \frac{\lambda}{\sinh(\lambda t)} \right]^{\frac{4-\delta}{2}}, \quad (9)$$

where  $\delta < 2$ ,  $\lambda > 0$ ,  $x > 0$  and  $\nu = \frac{4-\delta}{2} - 1$ .

We have for  $0 \leq y \leq x$

$$T_{x \rightarrow 0} \stackrel{(\text{law})}{=} T_{x \rightarrow y} + T_{y \rightarrow 0},$$

where  $T_{x \rightarrow y}$  and  $T_{y \rightarrow 0}$  are independent because of the strong Markov property. Hence, with (9) we obtain for the Laplace transform (LT) of  $T_{x \rightarrow y}$

$${}^{-\lambda}E_x^\delta [\exp(-\mu T_{x \rightarrow y})] = \frac{\phi_x(\mu)}{\phi_y(\mu)}, \quad (10)$$

where

$$\phi_x(\mu) = {}^{-\lambda}E_x^\delta [\exp(-\mu T_{x \rightarrow 0})] = \int_0^\infty \exp(-\mu t) {}^{-\lambda}p_x^\delta(t) dt.$$

Our main results are Theorem 1, Theorem 2, Corollary 1 and Corollary 2, where we derive explicit expressions of the LTs of first hitting times by radial Ornstein–Uhlenbeck processes with arbitrary  $x, y \geq 0$ . But let us first concentrate on the case  $\lambda = 0$ , i.e. Bessel processes.

## 2.1 First hitting times of Bessel processes.

In order to find an expression of the LT of first hitting times of Bessel processes, it is convenient to consider Bessel processes time reversed. For time reversed diffusion processes see Appendix B. Let  $(X_t)$  be a Bessel process with dimension  $\delta < 2$  starting in  $x > 0$ . The time reversed process

$$\hat{X}_u \equiv X_{T_{x \rightarrow 0} - u}, \quad u \geq 0,$$

is a  $\hat{\delta}$ -dimensional Bessel process starting in 0, with  $\hat{\delta} \equiv (4 - \delta)$ , see Appendix B. The process  $(\hat{X}_u)$  is transient since  $\hat{\delta} > 2$ . We have:

$$(X_{T_{x \rightarrow 0} - u}, u \leq T_{x \rightarrow 0}) = (\hat{X}_u, u \leq \hat{L}_{0 \rightarrow x}),$$

where

$$\hat{L}_{0 \rightarrow x} = \sup\{t \mid \hat{X}_t = x\}$$

is the last exit time of  $x$  by the process  $\{\hat{X}_t\}$ . In particular, as remarked in Gettoor–Sharpe [26], Sharpe [53],  $\hat{L}_{0 \rightarrow x}$  under  $P_0^{\hat{\delta}}$  has the same law as  $T_{x \rightarrow 0}$  under  $P_x^\delta$ . We know (see Gettoor [25], Pitman–Yor [45]):

$$\hat{L}_{0 \rightarrow x} \stackrel{(\text{law})}{=} \frac{x^2}{2Z_{\hat{\nu}}}, \quad (11)$$

where  $Z_{\hat{\nu}}$  is a Gamma variable with parameter  $\hat{\nu} \equiv \frac{\hat{\delta}}{2} - 1$ , i.e.:

$$P(Z_{\hat{\nu}} \in dt) = \frac{t^{\hat{\nu}-1} e^{-t}}{\Gamma(\hat{\nu})} dt, \quad (12)$$

and hence

$$P_x^\delta(T_{x \rightarrow 0} \in dt) = P_0^{\hat{\delta}}(\hat{L}_{0 \rightarrow x} \in dt) = \frac{1}{t\Gamma(\hat{\nu})} \left(\frac{x^2}{2t}\right)^{\hat{\nu}} e^{-\frac{x^2}{2t}} dt. \quad (13)$$

We note that (9) was obtained in an analogous way, additionally using Girsanov’s transformation.

Since, for  $y \neq 0$ ,  $y < x$ , the first hitting time  $T_{x \rightarrow y}$  is not distributed as  $\hat{L}_{y \rightarrow x}$ , the last exit time of  $x$  by  $(\hat{X}_u)$  starting in  $y$ , we cannot use the same time

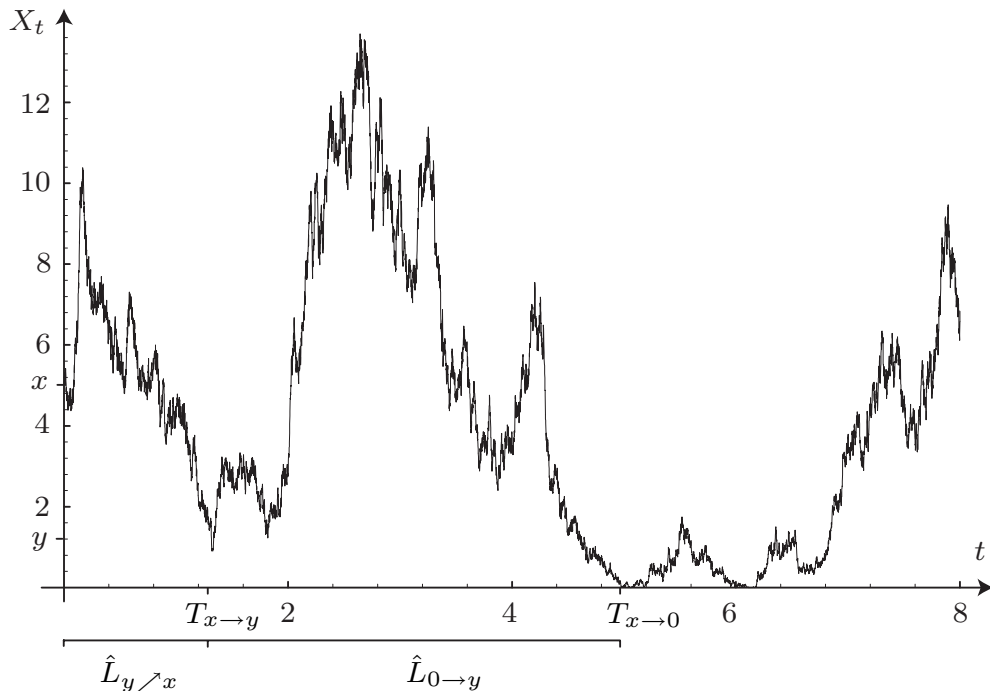


Figure 1: A simulated path of a  $\text{BES}_x^\delta$  process,  $\delta < 2$ , starting in  $x = 5$ .

reversal argument as above in order to find an expression for its LT.  $T_{x \rightarrow y}$  has the same law as  $\hat{L}_{y \nearrow x}$  where  $\hat{L}_{y \nearrow x}$  denotes the last exit time of  $x$  by  $(\hat{X}_u)$  starting in  $y$  at the last exit time of  $y$ , i.e., we consider the process starting in  $y$  conditioned on never visiting  $y$  again. We have

$$\hat{L}_{0 \rightarrow x} \stackrel{(\text{law})}{=} \hat{L}_{0 \rightarrow y} + \hat{L}_{y \nearrow x}, \quad (14)$$

where  $\hat{L}_{0 \rightarrow y}$  and  $\hat{L}_{y \nearrow x}$  are independent because of the strong Markov property (or path decomposition) at last exit times (for a survey see Millar [38]). Figure 1 illustrates the situation.

Our aim is to find an expression of the LT of  $\hat{L}_{y \nearrow x}$ :

$$E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} \hat{L}_{y \nearrow x} \right) \right] = \frac{E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} \hat{L}_{0 \rightarrow x} \right) \right]}{E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} \hat{L}_{0 \rightarrow y} \right) \right]}.$$

With (13) and with an integral representation of  $K_{\hat{\nu}}$ , the modified Bessel function of the third kind of order  $\hat{\nu}$ , (see e.g. Lebedev [35] p. 119, (5.10.25)) we



obtain the LT of  $\hat{L}_{0 \rightarrow y}$ :

$$E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} \hat{L}_{0 \rightarrow y} \right) \right] = \frac{(\alpha y)^{\hat{\nu}} K_{\hat{\nu}}(\alpha y)}{2^{\hat{\nu}-1} \Gamma(\hat{\nu})}.$$

Thus (see also Gettoor [25], Gettoor–Sharpe [26], Kent [32] (3.7) and Pitman–Yor [45]) for  $y < x$ ,  $y \neq 0$ :

$$\begin{aligned} E_x^\delta \left[ \exp \left( -\frac{\alpha^2}{2} T_{x \rightarrow y} \right) \right] &= E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} \hat{L}_{y \nearrow x} \right) \right] \\ &= \left( \frac{x}{y} \right)^{\hat{\nu}} \frac{K_{\hat{\nu}}(\alpha x)}{K_{\hat{\nu}}(\alpha y)} = \left( \frac{y}{x} \right)^\nu \frac{K_\nu(\alpha x)}{K_\nu(\alpha y)}. \end{aligned}$$

In the case  $0 < x < y$  we have, see Kent [32] (3.8):

$$E_x^\delta \left[ \exp \left( -\frac{\alpha^2}{2} T_{x \rightarrow y} \right) \right] = \left( \frac{y}{x} \right)^\nu \frac{I_\nu(\alpha x)}{I_\nu(\alpha y)},$$

with  $I_\nu$  the modified Bessel function of the first kind of order  $\nu$ . Since

$$\lim_{\varepsilon \downarrow 0} \frac{I_\nu(\mu\varepsilon)}{\varepsilon^\nu} = \left( \frac{\mu}{2} \right)^\nu \Gamma^{-1}(\nu + 1) \quad (15)$$

we obtain for  $x = 0$ ,  $y > 0$

$$E_0^\delta \left[ \exp \left( -\frac{\alpha^2}{2} T_{0 \rightarrow y} \right) \right] = \left( \frac{\alpha}{2} \right)^\nu \Gamma^{-1}(\nu + 1) \frac{y^\nu}{I_\nu(\alpha y)}.$$

We remark that for Brownian motion, that is, in the case  $\delta = 1$ , we have

$$T_{x \rightarrow 0} \stackrel{(\text{law})}{=} T_{y \rightarrow 0} + (\tilde{T}_{(x-y) \rightarrow 0}),$$

where  $x > y \geq 0$  and  $(\tilde{T}_{(x-y) \rightarrow 0})$  denotes an independent copy of  $T_{x \rightarrow y}$ . We know the density of  $T_{x \rightarrow 0}$  for Brownian motion

$$q_x(t) = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}},$$

i. e., we know the density of  $T_{(x-y) \rightarrow 0}$  and hence the density of  $T_{x \rightarrow y}$ . Note that

$$E_x^1 \left[ \exp \left( -\frac{\alpha^2}{2} T_{x \rightarrow 0} \right) \right] = \int_0^\infty e^{-\frac{\alpha^2 t}{2}} q_x(t) dt = e^{-\alpha x}.$$

## 2.2 First hitting times of radial Ornstein–Uhlenbeck processes.

Our aim is to find an explicit expression of the LT (10) of  $T_{x \rightarrow y}$  of a radial Ornstein–Uhlenbeck process  $R$ . We pursue an idea we got by studying Breiman

[7] and exploit the relation between CIR and BES processes. Breiman considers an Ornstein–Uhlenbeck process  $Y(u) = e^{-u}B(e^{2u})$  with  $B$  standard Brownian motion,  $B(1) = 0$ , and gives the LT of the first hitting time  $T_c$  for  $Y$  from the boundaries  $\pm c$

$$E(e^{-\alpha T_c}) = \tilde{D}_\alpha^{-1}(c),$$

where

$$\tilde{D}_\alpha(c) = \frac{2^{1-\lambda/2}}{\Gamma(\frac{\lambda}{2})} \int_0^\infty e^{-\frac{t^2}{2}} t^{\lambda-1} \cosh(ct) dt,$$

see also Borodin and Salminen [6], II.7.2.0.1, p. 429. First hitting times of Ornstein–Uhlenbeck processes are also studied by Ricciardi and Sato [50] and Horowitz [29].

A  $\delta$ -dimensional radial Ornstein–Uhlenbeck process  $R_t$  with parameter  $-\lambda$ ,  $R_0 = 0$ , can be written as

$$R_t = e^{-\lambda t} X \left( \frac{e^{2\lambda t} - 1}{2\lambda} \right), \quad (16)$$

where  $X$  is a BES $^\delta$  process,  $X_0 = 0$ . We are interested in the first time  $T_x$ ,  $R_t$  hits the level  $x > 0$ . Analogously to Breiman [7] we assume that  $D_\alpha$  is a function such that  $\{D_\alpha(R_t)e^{-\alpha t}, t \geq 0\}$  is a martingale with respect to the filtration of  $X \left( \frac{e^{2\lambda t} - 1}{2\lambda} \right)$ . Equivalently,  $\{D_\alpha(\frac{1}{\sqrt{2\lambda u + 1}} X_u)(2\lambda u + 1)^{-\frac{\alpha}{2\lambda}}, u \geq 0\}$  is a martingale with respect to the filtration of  $X_u$ . This means via Itô's formula that  $\{D_\alpha(\frac{r}{\sqrt{2\lambda u + 1}})(2\lambda u + 1)^{-\frac{\alpha}{2\lambda}}, r \geq 0, u \geq 0\}$  solves  $\frac{1}{2} \frac{d^2}{dr^2} + \frac{\delta-1}{2r} \frac{d}{dr} + \frac{d}{du} = 0$ , i. e., it is a space-time harmonic function with respect to  $\frac{1}{2} \frac{d^2}{dr^2} + \frac{\delta-1}{2r} \frac{d}{dr} + \frac{d}{du}$ . Further we assume that  $(D_\alpha(r), r \leq \rho)$  is bounded for every  $\rho$ . Then  $\{D_\alpha(R_t)e^{-\alpha t}, t < T_x\}$  is bounded. Applying the optional stopping theorem (see e. g. Revuz–Yor [49] §2.3) we have

$$-^\lambda E_0^\delta[D_\alpha(R_{T_x})e^{-\alpha T_x}] = D_\alpha(0),$$

thus

$$-^\lambda E_0^\delta[e^{-\alpha T_x}] = \frac{D_\alpha(0)}{D_\alpha(x)}.$$

Now we are motivated to find the function  $D_\alpha$  explicitly. First, we derive the space-time harmonic functions for Bessel processes.

We know by Widder [59], Theorem 14.1.1, that the most general  $\mathbb{R}^+$ -valued space-time harmonic function  $h(x, t)$  with  $x \in \mathbb{R}^n$ ,  $t \geq 0$ , with respect to  $\frac{1}{2}\Delta_x + \frac{d}{dt}$  is of the form

$$h(x, t) = \int_{\mathbb{R}^n} \exp\left(\xi \cdot x - \frac{|\xi|^2 t}{2}\right) m(d\xi), \quad (17)$$

where  $m$  is a positive measure. We assume  $h$  is not identically zero. From (17) we obtain the general positive space-time harmonic function  $(\hat{h}(r, t), r, t \in \mathbb{R}_+)$

with respect to  $\frac{1}{2} \frac{d^2}{dr^2} + \frac{n-1}{2r} \frac{d}{dr} + \frac{d}{dt}$  via

$$\tilde{h}(r, t) = \int_{S^{n-1}} h(\theta r, t) \sigma(d\theta) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \exp(\xi \cdot \theta r - \frac{|\xi|^2 t}{2}) \sigma(d\theta) m(d\xi), \quad (18)$$

where  $\sigma$  is the uniform probability measure on the unit sphere  $S^{n-1}$ . We know the following integral representation for the Bessel function  $I_\nu$

$$I_\nu(|\eta|) = \frac{|\frac{\eta}{2}|^\nu}{\Gamma(\nu+1)} \int_{S^{n-1}} \exp(\eta \cdot \theta) \sigma(d\theta),$$

$\eta \in \mathbb{R}^n$ ,  $\nu = \frac{n}{2} - 1$ , see Lebedev [35], §5.10, and (5.c2) in Pitman–Yor [45], and inserting in (18) we have

$$\tilde{h}(r, t) = \Gamma(\nu+1) 2^\nu \int_{\mathbb{R}^n} I_\nu(|\xi|r) \exp\left(-\frac{|\xi|^2 t}{2}\right) (|\xi|r)^{-\nu} m(d\xi).$$

Substituting  $|\xi|$  by  $u \in \mathbb{R}_+$  we obtain a space time harmonic function  $\tilde{h}$  with respect to a Bessel process with index  $\nu$

$$\tilde{h}(r, t) = \Gamma(\nu+1) 2^\nu \int_0^\infty I_\nu(ur) e^{-\frac{u^2 t}{2}} (ur)^{-\nu} \tilde{m}(du).$$

We believe this formula is true for all dimensions, i.e. we assume it for all indices  $\nu > -1$ . In fact, our aim is only to find a function  $D_\alpha$  such that

$$\begin{aligned} D_\alpha \left( \frac{r}{\sqrt{2\lambda t + 1}} \right) (2\lambda t + 1)^{-\frac{\alpha}{2\lambda}} &= \tilde{h}(r, t) \\ &= \Gamma(\nu+1) 2^\nu \int_0^\infty I_\nu(ur) e^{-\frac{u^2 t}{2}} r^{-\nu} f(u) du \end{aligned}$$

with an  $L^1$ -function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ; then  $\{D_\alpha(R_t) e^{-\alpha t}, t \leq T_x\}$  is a bounded martingale and the optional stopping theorem applies.

Let us first look for:

$$D_\alpha \left( \frac{r}{\sqrt{2\lambda t}} \right) (2\lambda t)^{-\frac{\alpha}{2\lambda}} = \Gamma(\nu+1) 2^\nu \int_0^\infty I_\nu(ru) e^{-\frac{u^2 t}{2}} r^{-\nu} f(u) du,$$

equivalently,

$$\begin{aligned} D_\alpha(y) &= (2\lambda t)^{\frac{\alpha}{2\lambda}} \Gamma(\nu+1) \left( \frac{2}{\sqrt{2\lambda t} y} \right)^\nu \int_0^\infty I_\nu(\sqrt{2\lambda t} y u) e^{-\frac{u^2 t}{2}} f(u) du \\ &= (2\lambda t)^{\frac{\alpha-\lambda}{2\lambda}} \Gamma(\nu+1) \left( \frac{2}{\sqrt{2\lambda t} y} \right)^\nu \int_0^\infty I_\nu(\eta y) e^{-\frac{\eta^2}{4\lambda}} f\left(\frac{\eta}{\sqrt{2\lambda t}}\right) d\eta. \quad (19) \end{aligned}$$

The righthandside of (19) should not depend on  $t$ , hence we may choose

$$f(x) \equiv c_0 x^{\frac{\alpha-\lambda}{\lambda}-\nu}, \quad (20)$$

with a constant  $c_0 > 0$ . We see that  $D_\alpha\left(\frac{r}{\sqrt{2\lambda t}}\right)(2\lambda t)^{-\frac{\alpha}{2\lambda}} = \tilde{h}(r, t)$ , and certainly also  $\tilde{h}(r, t + c)$  with a constant  $c$ , is a space-time harmonic function with respect to  $\frac{1}{2}\frac{d^2}{dr^2} + \frac{\delta-1}{2r}\frac{d}{dr} + \frac{d}{dt}$ . Thus

$$\begin{aligned} D_\alpha\left(\frac{r}{\sqrt{2\lambda t + 1}}\right) (2\lambda t + 1)^{-\frac{\alpha}{2\lambda}} \\ = \Gamma(\nu + 1) \left(\frac{2}{r}\right)^\nu \int_0^\infty I_\nu(ru) e^{-\frac{u^2}{2}(t+\frac{1}{2\lambda})} f(u) du \end{aligned}$$

with  $f$  in (20) and we find

$$D_\alpha(y) = \Gamma(\nu + 1) \left(\frac{2}{y}\right)^\nu \int_0^\infty I_\nu(\eta y) e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda} - \nu - 1} d\eta.$$

With (15) we see

$$D_\alpha(0) = \int_0^\infty e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda} - 1} d\eta = 2^{\frac{\alpha}{\lambda} - 1} \lambda^{\frac{\alpha}{2\lambda}} \Gamma\left(\frac{\alpha}{2\lambda}\right).$$

With the confluent hypergeometric function  $\phi$

$$\phi(a, b; z) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{z^j}{j!},$$

where  $b \neq 0, -1, -2, \dots$  and

$$(r)_0 = 1, \quad (r)_j = \frac{\Gamma(r + j)}{\Gamma(r)} = r(r + 1) \dots (r + j - 1),$$

$j = 1, 2, \dots$ , see e.g. Lebedev [35] (9.9.1), and with Whittaker's functions

$$M_{k, \mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{1}{2}z} \phi\left(\mu - k + \frac{1}{2}; 2\mu + 1; z\right),$$

see e.g. Lebedev [35] (9.13.16), finally, we obtain together with Bateman Manuscript Project [4] 4.16.(20)

**Theorem 1** *The LT of the first time  $T_x = \inf\{t \mid R_t = x\}$  a  $\delta$ -dimensional radial Ornstein-Uhlenbeck process  $R_t$  starting in 0 with parameter  $-\lambda$  hits the level  $x$  is*

$$\begin{aligned} -\lambda E_0^\delta(e^{-\alpha T_x}) &= \frac{2^{\frac{\alpha}{\lambda} - \nu - 1} x^\nu \Gamma\left(\frac{\alpha}{2\lambda}\right) \lambda^{\frac{\alpha}{2\lambda}}}{\Gamma(\nu + 1) \int_0^\infty I_\nu(\eta x) e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda} - \nu - 1} d\eta} \\ &= \frac{(\sqrt{\lambda}x)^{\nu+1} e^{-\frac{\lambda x^2}{2}}}{M_{\frac{1}{2}\left(-\frac{\alpha}{\lambda} + \nu + 1\right), \frac{\nu}{2}}(\lambda x^2)} \\ &= \frac{1}{\phi\left(\frac{\alpha}{2\lambda}, \nu + 1; \lambda x^2\right)}. \end{aligned}$$

For  $0 < y < x$  we have

$$T_{0 \rightarrow x} = T_{0 \rightarrow y} + T_{y \rightarrow x}$$

where  $T_{0 \rightarrow y}$  and  $T_{y \rightarrow x}$  are independent because of the strong Markov property. We deduce

**Corollary 1** *The LT of the first time  $T_x = \inf\{t \mid R_t = x\}$  a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process  $R_t$  starting in  $y$ ,  $0 < y < x$ , with parameter  $-\lambda$  hits the level  $x$  is*

$$\begin{aligned} -\lambda E_y^\delta(e^{-\alpha T_x}) &= \left(\frac{x}{y}\right)^{\nu+1} e^{\frac{\lambda}{2}(y^2-x^2)} \frac{M_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1), \frac{\nu}{2}}(\lambda y^2)}{M_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1), \frac{\nu}{2}}(\lambda x^2)} \\ &= \frac{\phi(\frac{\alpha}{2\lambda}, \nu+1; \lambda y^2)}{\phi(\frac{\alpha}{2\lambda}, \nu+1; \lambda x^2)}. \end{aligned}$$

Analogously, we obtain the LT of  $T_x$  in the case  $0 < x < y$ . Note that here we have to use the modified Bessel functions of the third kind  $K_\nu$  instead of the modified Bessel functions of the first kind  $I_\nu$  such that the required uniform integrability assumption is satisfied. With the confluent hypergeometric function of the second kind  $\psi$

$$\psi(a, b; z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \phi(a, b; z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \phi(1+a-b, 2-b; z),$$

see e.g. Lebedev [35] (9.10.3), and with Whittaker's functions

$$W_{k, \mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} \psi(\mu-k+\frac{1}{2}; 2\mu+1; z),$$

see e.g. Lebedev [35] (9.13.16), we obtain with Bateman Manuscript Project [4] 4.16.(37)

**Theorem 2** *The LT of the first time  $T_x = \inf\{t \mid R_t = x\}$  a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process  $R_t$  starting in  $y$ ,  $0 < x < y$ , with parameter  $-\lambda$  hits the level  $x$  is*

$$\begin{aligned} -\lambda E_y^\delta(e^{-\alpha T_x}) &= \left(\frac{x}{y}\right)^\nu \frac{\int_0^\infty K_\nu(\eta y) e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda}-\nu-1} d\eta}{\int_0^\infty K_\nu(\eta x) e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda}-\nu-1} d\eta} \\ &= \left(\frac{x}{y}\right)^{\nu+1} e^{\frac{\lambda}{2}(y^2-x^2)} \frac{W_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1), \frac{\nu}{2}}(\lambda y^2)}{W_{\frac{1}{2}(-\frac{\alpha}{\lambda}+\nu+1), \frac{\nu}{2}}(\lambda x^2)} \\ &= \frac{\psi(\frac{\alpha}{2\lambda}, \nu+1; \lambda y^2)}{\psi(\frac{\alpha}{2\lambda}, \nu+1; \lambda x^2)}. \end{aligned}$$

With

$$K_\nu(x) \sim \frac{\Gamma(-\nu)}{2^{\nu+1}} x^\nu$$

for  $x \rightarrow 0$  and  $\nu < 0$ , we obtain immediately

**Corollary 2** *The LT of the first time  $T_0 = \inf\{t \mid R_t = 0\}$  a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process  $R_t$  starting in  $y$  with  $\delta < 2$  with parameter  $-\lambda$  hits 0 is*

$$\begin{aligned} -\lambda E_y^\delta(e^{-\alpha T_0}) &= \frac{2^{\nu - \frac{\alpha}{2\lambda} + 2} y^{-\nu} \lambda^{-\frac{\alpha}{2\lambda}}}{\Gamma(\frac{\alpha}{2\lambda}) \Gamma(-\nu)} \int_0^\infty K_\nu(\eta y) e^{-\frac{\eta^2}{4\lambda}} \eta^{\frac{\alpha}{\lambda} - \nu - 1} d\eta \\ &= 4(\sqrt{\lambda}y)^{-\nu-1} e^{\frac{\lambda y^2}{2}} \frac{\Gamma(\frac{\alpha}{2\lambda} - \nu)}{\Gamma(-\nu)} W_{\frac{1}{2}(-\frac{\alpha}{\lambda} + \nu + 1), \frac{\nu}{2}}(\lambda y^2) \\ &= 4 \frac{\Gamma(\frac{\alpha}{2\lambda} - \nu)}{\Gamma(-\nu)} \psi\left(\frac{\alpha}{2\lambda}, \nu + 1; \lambda y^2\right) \end{aligned}$$

For  $\delta \geq 2$  we have  $T_0 = \infty$  almost surely, i.e.,  $-\lambda E_y^\delta(e^{-\alpha T_0}) = 0$ .

**Remark 2** In the case treated in Corollary 2 we know the density of  $T_0$  by (9).

**Remark 3** As mentioned in Remark 1, we obtain immediately the Laplace transforms of first hitting times of *squared*  $\delta$ -dimensional Bessel or radial Ornstein–Uhlenbeck processes. Moreover, from the investigation of the behaviour of squared Bessel or radial Ornstein–Uhlenbeck processes with negative dimensions or negative starting points, see the introduction and Section 3, we have no difficulties in extending the results above to, e. g., the case  $x > 0 > y$ .

**Remark 4** In disguised versions the formulae above also appear in Pitman–Yor [45], Chapter 12, and Eisenbaum [19].

**Remark 5** We mention another approach to obtain results concerning first hitting times of radial Ornstein–Uhlenbeck processes. Consider a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process  $(R_t)$ , starting in  $x > 0$ ,  $\delta < 2$ . We are interested in  $T_y = \inf\{t \mid R_t = y\}$ ,  $x > y \geq 0$ . In analogy to the case of BES processes in Section 2.1 we consider the process  $(R_t)$  time reversed, starting in  $T_0 = \inf\{t \mid R_t = 0\}$ , see Appendix B. Denote the time reversed process by  $(\hat{R}_t)$ . With the same notation as in (14) we are interested in  $\hat{L}_{x \nearrow y}$ , i. e., in the process  $(\hat{R}_t)$  after  $\hat{L}_{0 \rightarrow y}$ . By using a specific technique, the “enlargement of filtration“, we can write  $(\hat{R}_t)$  after  $\hat{L}_{0 \rightarrow y}$  as a diffusion. For a treatment see e.g. Jeulin [31] and Yor [66] §12. Heuristically spoken, we enlarge the original filtration progressively, so that the last exit time  $\hat{L}_{0 \rightarrow y}$  becomes a stopping time. Applying Theorem 12.4 in Yor [66] we obtain

**Proposition 1** *For the diffusion process  $\tilde{R}_u \equiv \hat{R}_{(\hat{L}_{0 \rightarrow y} + u)}$  we have*

$$\tilde{R}_u = y + \int_0^u b(\tilde{R}_v) dv - \int_0^u \frac{s'(\tilde{R}_v)}{s(y) - s(\tilde{R}_v)} 1_{(\tilde{R}_v > y)} dv + \tilde{W}_u, \quad (21)$$

where  $u \geq 0$ ,  $b$  is the drift and  $s$  is the scale function of the transient diffusion  $\hat{R}$ .

As an illustration consider the process  $\hat{R}$  to be a transient BES process  $\hat{X}$ , i.e. a BES process with index  $\nu > 0$ , started in 0. Its scale function may be chosen as  $s(x) = -x^{-2\nu}$  and we obtain from (21)

$$\tilde{X}_u \equiv \hat{X}_{(\hat{L}_0 \rightarrow y+u)} = y + \int_0^u \frac{1}{\tilde{X}_v} \frac{(\nu + \frac{1}{2})\tilde{X}_v^{2\nu} + (\nu - \frac{1}{2})y^{2\nu}}{\tilde{X}_v^{2\nu} - y^{2\nu}} dv + \tilde{W}_u. \quad (22)$$

For a BES<sup>3</sup>(0) process  $\hat{X}$  (22) reduces to

$$\tilde{X}_u \equiv \hat{X}_{(\hat{L}_0 \rightarrow y+u)} = y + \int_0^u \frac{dv}{\tilde{X}_v - y} + \tilde{W}_u.$$

Hence for a BES<sup>3</sup>(0) process  $X$  we have

$$(X_{L_y+u} - y, u \geq 0) \stackrel{(\text{law})}{=} (X_u, u \geq 0).$$

In general, the transition density  $\tilde{p}$  of the diffusion  $\tilde{R}$  in (21) is unknown. Note that if it were known, we would obtain the density of the last hitting time of  $x$  by the process  $\tilde{R}$  immediately from the formula

$$P_y(\tilde{L}_x \in dt) = -\frac{1}{2s(x)} \tilde{p}_t(y, x) dt,$$

see Borodin–Salminen [6] IV.43, Revuz–Yor [49] VII.(4.16), where  $\tilde{R}_0 = y$  and  $s$  is the scale function of  $\tilde{R}$  with  $\lim_{a \downarrow 0} s(a) = -\infty$  and  $s(\infty) = 0$ .

**Remark 6** As motivated in the end of Section 1 we are interested in the resolvent of radial Ornstein–Uhlenbeck processes. We obtain for the resolvent (see (6)) with  $f(r) \equiv (r - k)^+$

$$E_x[\int_0^\infty e^{-\alpha t} f(R_t) dt] = \int_0^\infty f(r) \int_0^\infty e^{-\alpha t} p_t(x, r) dt dr$$

with  $p_t$  the transition density of  $R_t$ , leaving us with the computation of  $\int_0^\infty e^{-\alpha t} p_t(x, y) dt$ . First, we consider BES <sub>$x$</sub>  <sup>$\delta$</sup>  processes and obtain with Appendix A.2

$$\int_0^\infty e^{-\alpha t} \frac{1}{t} y \left(\frac{y}{x}\right)^\nu \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right) dt = 2 \left(\frac{y}{x}\right)^\nu y I_\nu(z_1) K_\nu(z_2),$$

where

$$z_1 = \sqrt{2\alpha} \min(x, y)$$

and

$$z_2 = \sqrt{2\alpha} \max(x, y),$$

see Oberhettinger–Badii [44] (I.1.15.55). For a treatment of the resolvent for Bessel processes we refer to Itô–McKean [30], and also to De Schepper–Goovaerts–Delbaen [11] and Alili–Gruet [1]. Using relation (16) we obtain for a  $\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $-\lambda$

$$\begin{aligned} & \int_0^\infty e^{-\alpha t} 2y \lambda \frac{e^{2\lambda t}}{e^{2\lambda t} - 1} \left( \frac{e^{\lambda t} y}{x} \right)^\nu \exp \left( -\lambda \frac{x^2 + e^{2\lambda t} y^2}{e^{2\lambda t} - 1} \right) I_\nu \left( \frac{2\lambda x y e^{\lambda t}}{e^{2\lambda t} - 1} \right) dt \\ &= y \left( \frac{y}{x} \right)^\nu \int_0^\infty \frac{(u+1)^{\frac{1}{2}(\nu - \frac{\alpha}{\lambda})}}{u} \exp \left( -\lambda \frac{x^2 + (u+1)y^2}{u} \right) I_\nu \left( \frac{2\lambda x y \sqrt{u+1}}{u} \right) du. \end{aligned}$$

### 3 BESQ processes with negative dimensions and extensions

Bessel processes with nonnegative dimension  $\delta \geq 0$  and starting point  $x \geq 0$  are well-studied, e.g. Revuz–Yor [49], Chapter XI. As already pointed out in the introduction it seems to be quite natural also to consider Bessel processes with negative dimensions or negative starting points. Therefore we are motivated to extend the definition of  $\text{BESQ}_x^\delta$  processes and to introduce the class of  $\text{BESQ}_x^\delta$  processes with arbitrary  $\delta, x \in \mathbb{R}$ .

**Definition 3** *The solution to the stochastic differential equation*

$$dX_t = \delta dt + 2 \sqrt{|X_t|} dW_t, \quad X_0 = x, \quad (23)$$

where  $\{W_t\}$  is a one-dimensional Brownian motion,  $\delta \in \mathbb{R}$  and  $x \in \mathbb{R}$ , is called the square of a  $\delta$ -dimensional Bessel process, starting in  $x$ , and is denoted by  $\text{BESQ}_x^\delta$ .

Moreover, we generalize the definition of a CIR process, i.e., a squared  $\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $-\lambda$  in (7), by allowing the starting point and  $\delta$  to be in  $\mathbb{R}$ .

**Definition 4** *The solution to the stochastic differential equation*

$$dX_t = (\delta + 2\lambda X_t) dt + 2 \sqrt{|X_t|} dW_t, \quad X_0 = x, \quad (24)$$

where  $\delta, \lambda, x \in \mathbb{R}$  and  $\{W_t\}$  is a one-dimensional Brownian motion, is called a squared  $\delta$ -dimensional radial Ornstein–Uhlenbeck process with parameter  $\lambda$ .

First, we will derive and discuss properties of  $\text{BESQ}_x^\delta$  processes (23) with  $\delta, x \in \mathbb{R}$ , and as an extension we study CIR processes (24) in Section 3.1.

As mentioned in the introduction equation (23) is not the only possible way of defining  $\text{BESQ}_x^\delta$  processes with  $\delta \in \mathbb{R}$ . We will discuss this point after investigating the behaviour of  $\text{BESQ}_x^\delta$  processes with  $\delta \in \mathbb{R}$  defined by (23). Equation



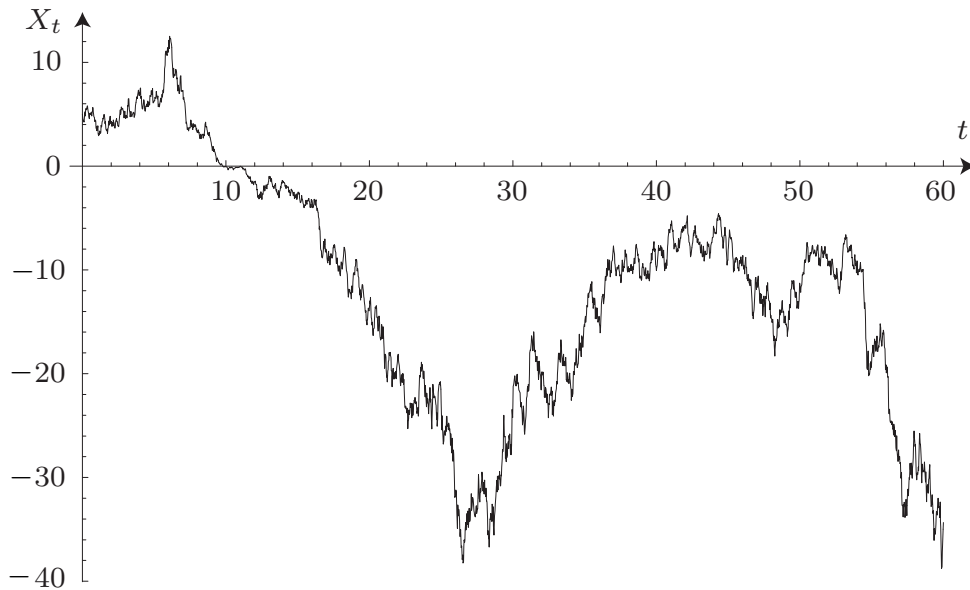


Figure 2: A simulated path of a  $\text{BES}^{-\gamma}$  process with  $\gamma \geq 2$ ,  $x_0 = 5$ .

(23) has a unique strong solution (see Revuz–Yor [49], Chapter IX §3). Denote its law by  $Q_x^\delta$ . First, we want to investigate the behaviour of a  $\text{BESQ}_x^\delta$  process, starting in  $x > 0$  with dimension  $\delta \leq 0$ . In the case  $\delta = 0$  the process reaches 0 in finite time and stays there. As for the case  $\delta < 0$ , we deduce from the comparison theorem that 0 is also reached in finite time. Let us consider the behaviour of a  $\text{BESQ}_x^\delta$  process  $\{X_t\}$  with  $\delta < 0$  and  $x > 0$  after it reached 0; we find:

$$\tilde{X}_u \equiv X_{T_0+u} = \delta u + 2 \int_{T_0}^{T_0+u} \sqrt{|X_s|} dW_s, \quad u \geq 0, \quad (25)$$

where  $T_0 = \inf\{t \mid X_t = 0\}$  denotes the first time the process  $\{X_t\}$  hits 0. With the notation  $\gamma \equiv -\delta$  we obtain from (25)

$$-\tilde{X}_u = \gamma u + 2 \int_0^u \sqrt{|\tilde{X}_s|} d\tilde{W}_s, \quad u \geq 0,$$

where  $\tilde{W}_s \equiv -(W_{s+T_0} - W_{T_0})$ , that is, after the  $\text{BESQ}_x^{-\gamma}$  process  $\{X_t\}$  hits 0, it behaves as a  $-\text{BESQ}_0^\gamma$ ,  $\gamma > 0$ . Two simulated paths of a  $\text{BES}_x^{-\gamma}$  process with  $\gamma \geq 2$ , respectively  $\gamma < 2$ , are shown in Figures 2 and 3.

From the above discussion we deduce that a  $\text{BESQ}_x^\delta$  process with  $\delta < 0$  and  $x \leq 0$  behaves as a  $-\text{BESQ}_{-x}^{-\delta}$ , especially it never becomes positive. For a  $\text{BESQ}_x^\delta$  process with dimension  $\delta \geq 0$  and starting point  $x \leq 0$ , we obtain with the same

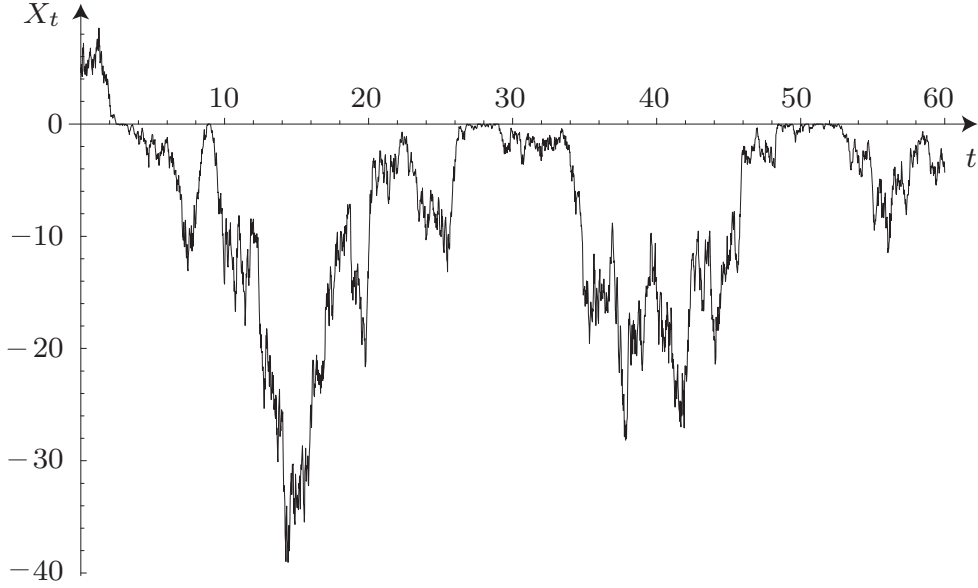


Figure 3: A simulated path of a  $\text{BES}^{-\gamma}$  process with  $\gamma < 2$ ,  $x_0 = 5$ .

argument, that it behaves as a  $-\text{BESQ}_{-x}^{-\delta}$  process; this means, until it hits 0 for the first time it behaves as a  $-\text{BESQ}_{-x}^{-\delta}$  process, and after that it behaves as a  $\text{BESQ}_0^\delta$ .

Let us now pursue another way of extending Definition 1 to  $\text{BESQ}^\delta$  processes with  $\delta \in \mathbb{R}$ . Instead of a  $\text{BESQ}^\delta$  process in (23) with diffusion coefficient  $\sigma(x) = 2\sqrt{|x|}$  we consider the process

$$X_t = x_0 - \delta t + 2 \int_0^t \sqrt{\alpha X_s^+ + \beta X_s^-} dB_s,$$

where  $\alpha, \beta \geq 0$ ,  $x^+ = \max(x, 0)$ ,  $x^- = \max(-x, 0)$ , and as before  $x_0 > 0$ ,  $\delta > 0$ . For  $t \leq T_0$ , we have

$$X_t = x_0 - \delta t + 2 \int_0^t \sqrt{\alpha X_s} dB_s. \quad (26)$$

After  $T_0$  the process evolves as

$$X_{T_0+t} = -\delta t + 2 \int_{T_0}^{T_0+t} \sqrt{\alpha X_s^+ + \beta X_s^-} dB_s,$$

and with  $Y_t \equiv -X_{T_0+t}$  we have

$$Y_t = \delta t + 2 \int_0^t \sqrt{\alpha Y_s^- + \beta Y_s^+} d\tilde{B}_s.$$

This admits only one solution which is the positive process

$$Y_t = \delta t + 2 \int_0^t \sqrt{\beta Y_s} d\tilde{B}_s. \quad (27)$$

Note that the parameter  $\beta$  in (27), as well as the parameter  $\alpha$  in (26), can be thought of as coming from a time transformation of a BESQ $^\gamma$  process  $Z$

$$Z_t = \gamma t + 2 \int_0^t \sqrt{Z_s} dB_s,$$

since

$$Z_{c^2 t} = \gamma c^2 t + 2 \int_0^t \sqrt{c^2 Z_{c^2 s}} d\tilde{B}_s.$$

After this remark let us now continue with investigating properties of processes defined by (23). An important and well-known property of squared Bessel processes with nonnegative dimensions is the additivity property, see Appendix A.4. The additivity property is no longer true for BESQ $^\delta$  processes with  $\delta \in \mathbb{R}$  arbitrary. Consider the BESQ $^\beta_x$  process  $Z^\beta$  and the BESQ $^{\tilde{\beta}}_y$  process  $Z^{\tilde{\beta}}$ , where  $\beta > 0$ ,  $\tilde{\beta} \equiv -\gamma < 0$  with  $\beta \geq \gamma$ ,  $x \geq 0$  and  $y \leq 0$ . Assuming the additivity property holds, would yield:

$$Z^\beta + Z^{\tilde{\beta}} \stackrel{(\text{law})}{=} Z^\beta - Z^\gamma \stackrel{(\text{law})}{=} Z^{\beta-\gamma} \geq 0,$$

since  $\beta \geq \gamma$ , which cannot be true because of the independence of  $Z^\beta$  and  $Z^\gamma$ . Our aim is to find the semigroup of a BESQ $^{-\gamma}_x$  process  $\{X_t\}$  with  $\gamma \geq 0$ ,  $x \geq 0$ . We have to decompose separately the process  $(X_t)$  before and after it hits 0 for the first time:

$$E_x^{-\gamma}[f(X_t)] = \underbrace{E_x^{-\gamma}[f(X_t) 1_{(t < T_0)}]}_A + \underbrace{E_x^{-\gamma}[f(X_t) 1_{(t \geq T_0)}]}_B.$$

Consider a BESQ $^{-\gamma}_x$  process  $\{X_t\}$  with  $\gamma \geq 0$ ,  $x \geq 0$ . Via time reversal we have

$$(X_t, t \leq T_0) \stackrel{(\text{law})}{=} (\tilde{X}_{\tilde{L}_x - t}, t \leq \tilde{L}_x), \quad (28)$$

where  $(\tilde{X}_t, t \geq 0)$  denotes a BESQ $^{4+\gamma}_0$  process and  $\tilde{L}_x = \sup\{t | \tilde{X}_t = x\}$ , see Appendix B. By means of (28) we obtain for the first term  $A$

$$A = E_0^{4+\gamma}[f(X_{\tilde{L}_x - t}) 1_{(t < \tilde{L}_x)}],$$

where  $X$  is a  $\text{BESQ}_0^{4+\gamma}$  process. Since we know, see (11),

$$\tilde{L}_x \stackrel{(\text{law})}{=} \frac{x}{2Z_\nu},$$

where  $Z_\nu$  is a Gamma variable with parameter  $\nu \equiv \frac{4+\gamma}{2} - 1$ , see (12), we have

$$Q_0^{4+\gamma}(\tilde{L}_x \in dt) = \frac{1}{t\Gamma(\nu)} \left(\frac{x}{2t}\right)^\nu e^{-\frac{x}{2t}} dt \equiv q_x(t) dt. \quad (29)$$

We obtain

$$\begin{aligned} E_0^{4+\gamma}[f(X_{\tilde{L}_x-t}) 1_{(t < \tilde{L}_x)}] &= \int_t^\infty E_0^{4+\gamma}[f(X_{s-t})] q_x(s) ds \\ &= \int_t^\infty \left( \int_0^\infty f(y) p_{s-t}^{4+\gamma}(0, y) dy \right) q_x(s) ds, \end{aligned}$$

where  $p_t^\delta(0, y)$ , with  $\delta > 0$ , is the semigroup density  $p_t^\delta(x, y)$  in  $y$  of  $\text{BESQ}^\delta$  for  $x = 0$

$$p_t^\delta(0, y) = \left(\frac{1}{2t}\right)^{\frac{\delta}{2}} \frac{1}{\Gamma(\frac{\delta}{2})} y^{\frac{\delta}{2}-1} \exp\left(\frac{-y}{2t}\right).$$

Analogously, for the second term  $B$  we have, since  $(-X_{t-T_0})$  is a  $\text{BESQ}_0^\gamma$ ,

$$B = \int_0^t \left( \int_0^\infty f(-y) p_{t-s}^\gamma(0, y) dy \right) q_x(s) ds.$$

Putting  $A$  and  $B$  together we obtain

$$\begin{aligned} E_x^{-\gamma}[f(X_t)] &= \int_t^\infty \left( \int_0^\infty f(y) p_{s-t}^{4+\gamma}(0, y) dy \right) q_x(s) ds \\ &\quad + \int_0^t \left( \int_0^\infty f(-y) p_{t-s}^\gamma(0, y) dy \right) q_x(s) ds, \end{aligned}$$

and hence, the semigroup density  $\hat{p}_t^{(-\gamma)}$  in  $y$  of a  $\text{BESQ}^{-\gamma}$  process is equal to

$$\hat{p}_t^{(-\gamma)}(x, y) = \int_t^\infty p_{s-t}^{4+\gamma}(0, y) q_x(s) ds,$$

for  $y > 0$ , and

$$\hat{p}_t^{(-\gamma)}(x, y) = \int_0^t p_{t-s}^\gamma(0, |y|) q_x(s) ds,$$

for  $y < 0$ . We want to find the semigroup density  $\hat{p}_t^{(-\gamma)}$  more explicitly. Consider the case  $y > 0$ :

$$\begin{aligned} \hat{p}_t^{(-\gamma)}(x, y) &= g(x, y, \gamma) \int_t^\infty (s(s-t))^{-(2+\frac{\gamma}{2})} \exp\left[-\frac{1}{2}\left[\frac{y}{s-t} + \frac{x}{s}\right]\right] ds \\ &= g(x, y, \gamma) \int_1^\infty \frac{t^{-3-\gamma}}{(u(u-1))^{(2+\frac{\gamma}{2})}} \exp\left[-\frac{1}{2}\left[\frac{y}{t(u-1)} + \frac{x}{tu}\right]\right] du, \end{aligned}$$

where

$$g(x, y, \gamma) = \Gamma^{-2} \left(1 + \frac{\gamma}{2}\right) \frac{(xy)^{1+\frac{\gamma}{2}}}{2^{(2+\gamma)}(2+\gamma)}.$$

With

$$h(x, y, \gamma, t) \equiv g(x, y, \gamma) t^{-3-\gamma}, \quad m \equiv 2 + \frac{\gamma}{2}, \quad \alpha \equiv \frac{y}{2t}, \quad \beta \equiv \frac{x}{2t} \quad (30)$$

we obtain

$$\begin{aligned} \hat{p}_t^{(-\gamma)}(x, y) &= h(x, y, \gamma, t) \int_1^\infty \frac{du}{(u(u-1))^m} \exp\left(-\left(\frac{\alpha}{u-1} + \frac{\beta}{u}\right)\right) \\ &= h(x, y, \gamma, t) \int_0^1 \frac{s^{2(m-1)}}{(1-s)^m} \exp\left(-\beta s - \frac{\alpha s}{1-s}\right) ds, \end{aligned}$$

and hence we have

**Proposition 2** *The semigroup density  $\hat{p}_t^{(-\gamma)}$  in  $y$  of a BESQ $^{-\gamma}$  process is equal to*

$$\hat{p}_t^{(-\gamma)}(x, y) = h(x, y, \gamma, t) e^{\alpha-\beta} \int_0^1 \frac{(1-w)^{2(m-1)}}{w^m} \exp\left(\beta w - \frac{\alpha}{w}\right) dw \quad (31)$$

for  $y > 0$  with  $h$  in (30). Analogously, in the case  $y < 0$  we have

$$\hat{p}_t^{(-\gamma)}(x, y) = k(x, y, \gamma, t) e^{-\alpha-\beta} \int_0^\infty \frac{(w+1)^{2m}}{w^m} \exp\left(-\beta w - \frac{\alpha}{w}\right) dw, \quad (32)$$

with

$$k(x, y, \gamma, t) \equiv \Gamma^{-2} \left(\frac{\gamma}{2}\right) \frac{2^{-\gamma}}{\gamma} x^{\frac{\gamma}{2}+1} |y|^{\frac{\gamma}{2}-1} t^{-\gamma-1},$$

and

$$m \equiv \frac{\gamma}{2}, \quad \alpha \equiv \frac{|y|}{2t}, \quad \beta \equiv \frac{x}{2t}.$$

To our knowledge, there exist no closed-form expressions for the integrals in formulae (31) and (32).

As an application of the time reversal result (28) we make some computations concerning the law  $Q_x^{-\gamma}$  of a BESQ $^{-\gamma}$  process  $(Z_t)_{t \geq 0}$  with  $\gamma, x > 0$ . Since  $(-Z_{T_0+t}, t \geq 0)$  is a BESQ $^{\gamma}_0$  process, independent of the past of  $(Z_t)$  up to  $T_0$ , we have

$$\begin{aligned} Q_x^{-\gamma} \left[ \exp\left(-\int_0^\infty Z_u f(u) du\right) \right] &= \int_0^\infty Q_x^{-\gamma} \left[ \exp\left(-\int_0^t Z_u f(u) du\right) \middle| T_0 = t \right] \\ &\quad \times Q_0^\gamma \left[ \exp\left(\int_0^\infty X_v f(t+v) dv\right) \right] Q_x^{-\gamma} [T_0 \in dt], \end{aligned}$$

with a BESQ $_0^\gamma$  process  $(X_v)_{v \geq 0}$  and a positive Borel function  $f$ . In the following, negative dimensional BESQ processes are denoted by  $(Z_t)$  and positive dimensional ones by  $(X_t)$ . From the time reversal result (28) we obtain

$$Q_x^{-\gamma} \left[ \exp \left( - \int_0^t Z_u f(u) du \right) \middle| T_0 = t \right] = Q_0^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(t-u) du \right) \middle| L_x = t \right]$$

and with the equalities

$$\begin{aligned} & Q_0^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(t-u) du \right) \middle| L_x = t \right] \\ &= Q_0^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(t-u) du \right) \middle| X_t = x \right] \\ &= Q_x^{4+\gamma} \left[ \exp \left( - \int_0^t X_{t-u} f(t-u) du \right) \middle| X_t = 0 \right] \\ &= Q_x^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(u) du \right) \middle| X_t = 0 \right] \end{aligned}$$

finally, we have with (29),

$$\begin{aligned} Q_x^{-\gamma} \left[ \exp \left( - \int_0^\infty Z_u f(u) du \right) \right] &= \int_0^\infty Q_x^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(u) du \right) \middle| X_t = 0 \right] \\ &\quad \times Q_0^\gamma \left[ \exp \left( \int_0^\infty X_v f(t+v) dv \right) \right] q_x(t) dt. \end{aligned} \quad (33)$$

Our aim is to determine

$$Q_x^{4+\gamma} \left[ \exp \left( - \int_0^t X_u f(u) du \right) \middle| X_t = 0 \right]$$

and

$$Q_0^\gamma \left[ \exp \left( \int_0^\infty X_v f(t+v) dv \right) \right]$$

more explicitly. We will need the following lemma which is easy to verify.

**Lemma 1** *Consider a function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  in  $L^1$  with  $h(x) \leq c$  for all  $x \in \mathbb{R}^+$  and continuous in a neighbourhood of 0. Then*

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \lambda e^{-\lambda y} h(y) dy = h(0).$$

Applying Lemma 1 we deduce that the ratio

$$\begin{aligned} & \frac{Q_x^\delta \left( \exp \left( -\lambda X_t - \int_0^t X_u f(u) du \right) \right)}{Q_x^\delta \left( \exp \left( -\lambda X_t \right) \right)} \\ &= \frac{\int_0^\infty e^{-\lambda y} q_t^\delta(x, y) Q_x^\delta \left( \exp \left( - \int_0^t X_u f(u) du \right) \middle| X_t = y \right) dy}{\int_0^\infty e^{-\lambda y} q_t^\delta(x, y) dy}, \end{aligned} \quad (34)$$

as  $\lambda$  tends to infinity, converges to

$$Q_x^\delta \left( \exp\left(-\int_0^t X_u f(u) du\right) | X_t = 0 \right).$$

Let us consider the numerator and denominator of ratio (34) separately. Via formula (1.h) in Pitman–Yor [47] and Theorem (1.7) in Chapter XI in Revuz–Yor [49] we know

**Lemma 2** *Consider a positive function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\int_0^n f(s) ds < \infty$  for all  $n$ . Then*

$$\begin{aligned} Q_x^\delta \left[ \exp\left(-\int_0^t X_u f(u) du - \frac{\lambda}{2} X_t\right) \right] \\ = (\psi_f'(t) + \lambda \psi_f(t))^{-\delta/2} \exp \frac{x}{2} \left[ \phi_f'(0) - \frac{(\phi_f' + \lambda \phi_f)(t)}{(\psi_f' + \lambda \psi_f)(t)} \right], \end{aligned} \quad (35)$$

where  $\phi_f$  is the unique solution of the Sturm-Liouville equation

$$\frac{1}{2} \phi_f''(s) = f(s) \phi_f(s),$$

with  $s \in (0, \infty)$ ,  $\phi_f(0) = 1$ , which is positive and non increasing, and  $\phi_f'(0)$  is the right derivative in 0, and

$$\psi_f(t) \equiv \phi_f(t) \int_0^t \frac{ds}{\phi_f^2(s)}.$$

Furthermore

$$Q_x^\delta \left( \exp\left(-\int_0^\infty X_u f(u) du\right) \right) = \phi_f(\infty)^{\delta/2} \exp \left( \frac{x}{2} \phi_f'(0) \right), \quad (36)$$

where  $\phi_f(\infty) \in [0, 1]$  is the limit at infinity.

From the formula following Corollary (1.3) in Chapter XI in Revuz–Yor [49] we know

$$Q_x^\delta \left( \exp\left(-\frac{\lambda}{2} X_t\right) \right) = (1 + \lambda t)^{-\frac{\delta}{2}} \exp \left( \frac{-\frac{\lambda}{2} x}{1 + \lambda t} \right),$$

and together with (35) we obtain

$$\begin{aligned} Q_x^\delta \left( \exp\left(-\int_0^t X_u f(u) du\right) | X_t = 0 \right) \\ = \left( \frac{t}{\psi_f(t)} \right)^{\delta/2} \exp \left( \frac{x}{2} \left( \phi_f'(0) - \frac{\phi_f(t)}{\psi_f(t)} + \frac{1}{t} \right) \right). \end{aligned} \quad (37)$$

Now we will determine  $Q_x^\delta(\exp(\int_0^\infty X_s f(t+s)ds))$  more explicitly. Note that the function

$$\phi_t(s) \equiv \frac{\phi_f(t+s)}{\phi_f(t)}$$

solves

$$\frac{1}{2}\phi_t''(s) = f(t+s)\phi_t(s),$$

and that formula (36) leads to

$$\begin{aligned} Q_x^\delta \left( \exp - \int_0^\infty X_s f(t+s)ds \right) &= \phi_t(\infty)^{\delta/2} \exp \left( \frac{x}{2} \phi_t'(0) \right) \\ &= \left( \frac{\phi_f(\infty)}{\phi_f(t)} \right)^{\delta/2} \exp \left( \frac{x}{2} \frac{\phi_f'(t)}{\phi_f(t)} \right). \end{aligned}$$

**Lemma 3** For a positive decreasing function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , such that

$$Q_x^\delta \left( \exp \int_0^\infty X_s f(s)ds \right) < \infty,$$

there exists a function  $\tilde{\phi}$  which is on  $(0, \infty)$  the unique solution of

$$\frac{1}{2}\tilde{\phi}''(s) = -f(s)\tilde{\phi}(s),$$

with  $\tilde{\phi}(0) = 1$  such that

$$Q_x^\delta \left( \exp \int_0^\infty X_u f(t+u)du \right) = \left( \frac{\tilde{\phi}(\infty)}{\tilde{\phi}(t)} \right)^{\delta/2} \exp \left( \frac{x}{2} \frac{\tilde{\phi}'(t)}{\tilde{\phi}(t)} \right).$$

Proof: We want to determine functions  $\alpha(t)$  and  $\beta(t)$  with

$$Q_x^\delta \left( \exp \int_0^\infty X_u f(t+u)du \right) = (\beta(t))^{\delta/2} \exp(x\alpha(t)). \quad (38)$$

Using the Markov property we have

$$\begin{aligned} Q_x^\delta \left( \exp \int_0^\infty X_u f(u)du | \mathcal{F}_t \right) \\ = \exp \left( \int_0^t X_u f(u)du \right) Q_{X_t}^\delta \left( \exp \int_0^\infty X_u f(t+u)du \right). \end{aligned}$$

Denote

$$h(x, t; \omega) \equiv (\beta(t))^{\delta/2} \exp(x\alpha(t)) \exp \left( \int_0^t X_u f(u)du \right).$$



Since  $Q_x^\delta (\exp \int_0^\infty X_u f(u) du | \mathcal{F}_t)$  is a martingale, with Itô's formula we see that the functions  $\alpha$  and  $\beta$  have to fulfill the following condition in terms of  $h$

$$\frac{\partial}{\partial t} h(x, t) + 2x \frac{\partial^2}{\partial x^2} h(x, t) + \delta \frac{\partial}{\partial x} h(x, t) = 0.$$

This implies

$$\alpha'(t) + f(t) + 2\alpha^2(t) = 0, \quad \alpha(t) + \frac{\beta'(t)}{2\beta(t)} = 0.$$

Hence

$$\alpha(t) = -\frac{\beta'(t)}{2\beta(t)}, \quad f(t) = \frac{\beta''(t)}{2\beta(t)} - \left( \frac{\beta'(t)}{\beta(t)} \right)^2.$$

Consider the function  $\tilde{\phi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with

$$\tilde{\phi}(s) \equiv \frac{\beta(0)}{\beta(s)}.$$

Then  $\tilde{\phi}$  is on  $(0, \infty)$  the unique solution of

$$\frac{1}{2} \tilde{\phi}''(t) = -f(t) \tilde{\phi}(t),$$

with  $\tilde{\phi}(0) = 1$ . We obtain

$$\alpha(t) = \frac{\tilde{\phi}'(t)}{2\tilde{\phi}(t)}.$$

From (38) we deduce by dominated convergence that  $\lim_{t \rightarrow \infty} \beta(t) = 1$ , and hence  $\tilde{\phi}(\infty) \equiv \lim_{t \rightarrow \infty} \tilde{\phi}(t) = \beta(0)$ .

Altogether we obtain for (33) with (37) and Lemma 3

**Theorem 3**

$$Q_x^{-\gamma} \left( \exp \left( - \int_0^\infty Z_u f(u) du \right) \right) = \frac{1}{\Gamma(\frac{\gamma}{2} + 1)} \left( \frac{x}{2} \right)^{\frac{\gamma}{2} + 1} \\ \times \int_0^\infty \left( \frac{1}{\psi_f(t)} \right)^{\frac{\gamma}{2} + 2} \left( \frac{\tilde{\phi}(\infty)}{\tilde{\phi}(t)} \right)^{\gamma/2} \exp \left( \frac{x}{2} \left( \phi_f'(0) - \frac{\phi_f(t)}{\psi_f(t)} \right) \right) dt,$$

with  $\phi_f$  and  $\psi_f$  from Lemma 2, and  $\tilde{\phi}$  from Lemma 3.

Example: Consider

$$f(u) \equiv \frac{\lambda^2}{2} 1_{[0, a]}(u), \quad a, \lambda > 0.$$

We know (see Pitman–Yor [46], p. 432 (2.m))

$$\begin{aligned} Q_x^{4+\gamma} \left( \exp\left(-\frac{\lambda^2}{2} \int_0^t X_u du\right) \middle| X_t = 0 \right) \\ = \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{\frac{4+\gamma}{2}} \exp\left(-\frac{x}{2t}(\lambda t \coth(\lambda t) - 1)\right), \end{aligned}$$

and hence we have

$$\begin{aligned} Q_x^{-\gamma} \left( \exp\left(-\frac{\lambda^2}{2} \int_0^a Z_t dt\right) \right) &= \int_0^a \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{\frac{4+\gamma}{2}} \exp\left(-\frac{x}{2t}(\lambda t \coth(\lambda t) - 1)\right) \\ &\quad \times Q_0^\gamma \left( \exp\left(\frac{\lambda^2}{2} \int_0^{a-t} X_v dv\right) \right) q_x(t) dt \\ &\quad + \int_a^\infty Q_x^{4+\gamma} \left( \exp\left(-\frac{\lambda^2}{2} \int_0^a X_u du\right) \middle| X_t = 0 \right) q_x(t) dt. \end{aligned}$$

We investigate more deeply

$$Q_{x \rightarrow 0}^{4+\gamma} \left( \exp\left(-\frac{\lambda^2}{2} \int_0^a X_u du\right) \right) \equiv Q_x^{4+\gamma} \left( \exp\left(-\frac{\lambda^2}{2} \int_0^a X_u du\right) \middle| X_t = 0 \right),$$

where a continuous process with law  $Q_{x \rightarrow 0}^{4+\gamma}$  is called the *Squared Bessel Bridge* from  $x$  to 0 over  $[0, t]$ , see e.g. Revuz–Yor [49], XI.3. For  $\mathcal{A} \in \mathcal{F}_a$  we have

$$\begin{aligned} Q_{x \rightarrow 0}^{4+\gamma}(\mathcal{A}) &= \lim_{y \downarrow 0} \frac{E_x^{4+\gamma} (1_{\mathcal{A}} 1_{[0,y]}(X_t))}{Q_x^{4+\gamma}(X_t \in [0, y])} \\ &= \lim_{y \downarrow 0} \frac{E_x^{4+\gamma} (1_{\mathcal{A}} E_x^{4+\gamma}(1_{[0,y]}(X_t) | \mathcal{F}_a))}{Q_x^{4+\gamma}(X_t \in [0, y])} \\ &= \lim_{y \downarrow 0} E_x^{4+\gamma} \left( 1_{\mathcal{A}} \frac{Q_{X_a}^{4+\gamma}(X_{t-a} \in [0, y])}{Q_x^{4+\gamma}(X_t \in [0, y])} \right), \end{aligned}$$

and

$$\lim_{y \downarrow 0} \frac{Q_{X_a}^{4+\gamma}(X_{t-a} \in [0, y])}{Q_x^{4+\gamma}(X_t \in [0, y])} = \frac{p_{t-a}^{4+\gamma}(X_a, 0)}{p_t^{4+\gamma}(x, 0)} \equiv \psi(a, X_a),$$

where  $p_t^\delta(x, y)$  is the transition density of BESQ $^\delta$ ,  $\delta > 0$ . From the proof of the theorem in 0.5 in Yor [65] we obtain

$$\lim_{y \downarrow 0} \frac{p_{t-a}^{4+\gamma}(X_a, y)}{p_t^{4+\gamma}(x, y)} = \left( \frac{t}{t-a} \right)^{\frac{4+\gamma}{2}} \exp\left(-\frac{X_a}{2(t-a)}\right) \exp\left(\frac{x}{2t}\right).$$

Formula (2.k) in Pitman–Yor [46] gives us

$$\begin{aligned} & Q_x^{4+\gamma} \left( \exp \left( -\frac{X_a}{2(t-a)} - \frac{\lambda^2}{2} \int_0^a X_u du \right) \right) \\ &= \left( \cosh(\lambda a) + \frac{1}{\lambda(t-a)} \sinh(\lambda a) \right)^{-\frac{4+\gamma}{2}} \exp \left( -\frac{\lambda x}{2} \cdot \frac{1 + \frac{\coth(\lambda a)}{\lambda(t-a)}}{\coth(\lambda a) + \frac{1}{\lambda(t-a)}} \right). \end{aligned}$$

Hence we have

$$\begin{aligned} Q_{x \xrightarrow{(t)} 0}^{4+\gamma} \left( \exp \left( -\frac{\lambda^2}{2} \int_0^a X_u du \right) \right) &= \left( \left( \frac{t-a}{t} \right) \cosh(\lambda a) + \frac{1}{\lambda t} \sinh(\lambda a) \right)^{-\frac{4+\gamma}{2}} \\ &\quad \cdot \exp \left( \frac{x}{2t} - \frac{\lambda x}{2} \frac{1 + \frac{\coth(\lambda a)}{\lambda(t-a)}}{\coth(\lambda a) + \frac{1}{\lambda(t-a)}} \right). \end{aligned}$$

If in addition  $\lambda(a-t) < \frac{\pi}{2}$ , we obtain

$$Q_0^\gamma \left( \exp \left( \frac{\lambda^2}{2} \int_0^{a-t} X_v dv \right) \right) = \cos(\lambda(a-t))^{-\frac{\gamma}{2}},$$

and finally,

$$\begin{aligned} & Q_x^{-\gamma} \left( \exp \left( -\frac{\lambda^2}{2} \int_0^a Z_t dt \right) \right) \\ &= \frac{1}{\Gamma(\frac{\gamma}{2}+1)} \left( \frac{x}{2} \right)^{\frac{\gamma}{2}+1} \left[ \lambda^{\frac{\gamma}{2}+2} \int_0^a \exp \left( -\frac{x}{2} \lambda \coth(\lambda t) \right) \frac{\cos(\lambda(a-t))^{-\frac{\gamma}{2}}}{\sinh(\lambda t)^{\frac{\gamma}{2}+2}} dt \right. \\ &\quad \left. + \int_0^\infty [\cosh(\lambda a) u + \frac{1}{\lambda} \sinh(\lambda a)]^{-\frac{\gamma}{2}-2} \exp \left( -\frac{\lambda x}{2} \frac{\lambda u + \coth(\lambda a)}{\lambda \coth(\lambda a) u + 1} \right) du \right]. \end{aligned}$$

### 3.1 Extension to squared radial Ornstein–Uhlenbeck processes.

As an extension to BESQ $_x^\delta$  processes in (23) with  $\delta, x \in \mathbb{R}$  we now investigate squared  $\delta$ -dimensional radial Ornstein–Uhlenbeck processes, i.e., CIR processes, defined by (24) with  $\delta, x \in \mathbb{R}$ . We consider the case  $\delta < 0$  and  $x > 0$ ; for  $\lambda = 0$  this corresponds to a BESQ $_x^\delta$  process with negative dimension  $\delta$ . We call  ${}^\lambda Q_x^\delta$  the law on  $C(\mathbb{R}_+, \mathbb{R})$  and, as before, we denote  $Q_x^\delta \equiv {}^0 Q_x^\delta$ . Via Girsanov transformation we obtain the relationship

$${}^\lambda Q_x^\delta |_{\mathcal{F}_t} = \exp \left( \lambda \int_0^t \sqrt{|X_s|} \operatorname{sgn}(X_s) dW_s - \frac{\lambda^2}{2} \int_0^t |X_s| ds \right) Q_x^\delta |_{\mathcal{F}_t}. \quad (39)$$

We may also write the stochastic integral  $\int_0^t \operatorname{sgn}(X_s) \sqrt{|X_s|} dW_s$  in a simpler form, since we have from Itô's formula:

$$|X_t| = |x| + \int_0^t \operatorname{sgn}(X_s) (\delta ds + 2 \sqrt{|X_s|} dW_s) + L_t^0(X),$$

where  $L_t^0(X)$  is the semimartingale local time of  $X$  in 0. For  $L_t^0(X)$  we obtain with Revuz–Yor [49], p. 212, Corollary 1.9)

$$\begin{aligned} L_t^0(X) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[0, \varepsilon]}(X_s) d\langle X, X \rangle_s = \lim_{\varepsilon \downarrow 0} \frac{4}{\varepsilon} \int_0^t |X_s| 1_{[0, \varepsilon]}(X_s) ds \\ &\leq \lim_{\varepsilon \downarrow 0} \left( 4 \int_0^t 1_{[0, \varepsilon]}(X_s) ds \right) = 0. \end{aligned}$$

Hence we have

$$\int_0^t \operatorname{sgn}(X_s) \sqrt{|X_s|} dW_s = \frac{1}{2} \left( |X_t| - |x| - \delta \int_0^t \operatorname{sgn}(X_s) ds \right).$$

Thus, (39) takes the form

**Lemma 4**

$${}^\lambda Q_x^\delta |_{\mathcal{F}_t} \equiv \exp \left( \frac{\lambda}{2} \left( |X_t| - |x| - \delta \int_0^t \operatorname{sgn}(X_s) ds \right) - \frac{\lambda^2}{2} \int_0^t |X_s| ds \right) Q_x^\delta |_{\mathcal{F}_t}.$$

Applying Lemma 4 we obtain the conditional expectation formula

$$\begin{aligned} {}^\lambda \hat{p}_t^\delta(x, y) &= \hat{p}_t^\delta(x, y) \exp \left( \frac{\lambda}{2} (|y| - |x|) \right) \\ &\quad \times Q_x^\delta \left( \exp \left( -\frac{\delta \lambda}{2} \int_0^t \operatorname{sgn}(X_s) ds - \frac{\lambda^2}{2} \int_0^t |X_s| ds \right) \middle| X_t = y \right) \end{aligned} \quad (40)$$

where  $\hat{p}_t^\delta$  denotes the semigroup density in  $y$  of a BESQ $^\delta$  process,  $\delta < 0$ , given by (31) for  $y > 0$ , respectively by (32) for  $y < 0$ .

Using the time-space transformation from a BESQ $^\delta$  process  $(X_t^\delta)$  to a squared radial Ornstein–Uhlenbeck process  $({}^\lambda X_t^\delta)$

$${}^\lambda X_t^\delta = e^{2\lambda t} X_t^\delta \left( \frac{1 - e^{-2\lambda t}}{2\lambda} \right),$$

we also have together with the relationship (40)

$${}^\lambda \hat{p}_t^\delta(x, y) = e^{-2\lambda t} \hat{p}_t^\delta \left( \frac{1 - e^{-2\lambda t}}{2\lambda} \right) (x, e^{-2\lambda t} y),$$

from which  ${}^\lambda \hat{p}_t^\delta(x, y)$  is obtained since  $\hat{p}_t^\delta(x, y)$  is known see (31), (32). Hence we obtain from (40)

**Theorem 4** For  $y > 0$  we have

$$\begin{aligned} & Q_x^\delta \left( \exp \left( -\frac{\lambda^2}{2} \int_0^t X_s ds \right) \middle| X_t = y \right) \\ &= \hat{p}_t^\delta \left( \frac{1-e^{-2\lambda t}}{2\lambda} \right) (x, e^{-2\lambda t} y) \exp \left( -2\lambda t - \frac{\lambda}{2} (y-x) + \frac{\delta \lambda t}{2} \right) / \hat{p}_t^\delta(x, y), \end{aligned}$$

with  $\hat{p}_t^\delta(x, y)$  given by (31). For  $y < 0$  we have

$$\begin{aligned} & Q_x^\delta \left( \exp \left( -\frac{\delta \lambda}{2} \int_0^t \operatorname{sgn}(X_s) ds - \frac{\lambda^2}{2} \int_0^t |X_s| ds \right) \middle| X_t = y \right) \\ &= \hat{p}_t^\delta \left( \frac{1-e^{-2\lambda t}}{2\lambda} \right) (x, e^{-2\lambda t} y) \exp \left( -2\lambda t - \frac{\lambda}{2} (|y| - |x|) \right) / \hat{p}_t^\delta(x, y), \end{aligned}$$

with  $\hat{p}_t^\delta(x, y)$  given by (32).

## Appendices

### A Some properties of Bessel processes

#### A.1

In addition to Definitions 1 and 2 we give explicit expressions for Bessel processes. For  $\delta > 1$  a  $\text{BES}_{x_0}^\delta$  process  $X_t$  satisfies  $E[\int_0^t (ds/X_s)] < \infty$  and is the solution to the equation

$$X_t = x_0 + \frac{\delta - 1}{2} \int_0^t \frac{ds}{X_s} + W_t. \quad (41)$$

For  $\delta \leq 1$  the situation is less simple. For  $\delta = 1$  the role of (41) is played by

$$X_t = |W_t| = \tilde{W}_t + L_t,$$

where  $\tilde{W}_t \equiv \int_0^t \operatorname{sgn}(W_s) dW_s$  is a standard Brownian motion, and  $L_t$  is the local time of Brownian motion. For a treatment of local times see e.g. Revuz–Yor [49], Chapter VI. For  $\delta < 1$  we have

$$X_t = x_0 + \frac{\delta - 1}{2} \text{p.v.} \int_0^t \frac{ds}{X_s} + W_t, \quad (42)$$

where the principal value is defined as

$$\text{p.v.} \int_0^t \frac{ds}{X_s} \equiv \int_0^\infty x^{\delta-2} (L_t^x - L_t^0) dx$$

and the family of local times  $(L_t^x, x \geq 0)$  is defined as

$$\int_0^t \varphi(X_s) ds = \int_0^\infty \varphi(x) L_t^x x^{\delta-1} dx$$

for all Borel functions  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , see Bertoin [5]. The decomposition (42) was obtained using the fact that a power of a Bessel process is another Bessel process time-changed:

$$qX_\nu^{1/q}(t) = X_{\nu q} \left( \int_0^t \frac{ds}{X_\nu^{2/p}(s)} \right),$$

where  $\frac{1}{p} + \frac{1}{q} = 1, \nu > -\frac{1}{q}$ , see e.g. Revuz–Yor [49], Proposition 11.(1.11).

## A.2 Transition densities.

(Squared) Bessel processes are Markov processes and their transition densities are known explicitly. For  $\delta > 0$ , the transition density for  $\text{BESQ}^\delta$  is equal to

$$q_t^\delta(x, y) = \frac{1}{2t} \left( \frac{y}{x} \right)^{\frac{\nu}{2}} \exp \left\{ -\frac{x+y}{2t} \right\} I_\nu \left( \frac{\sqrt{xy}}{t} \right), \quad (43)$$

where  $t > 0, x > 0, \nu \equiv \frac{\delta}{2} - 1$  and  $I_\nu$  is the modified Bessel function of the first kind of index  $\nu$ . For  $x = 0$  we have

$$q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma(\delta/2)^{-1} y^{\frac{\delta}{2}-1} \exp \left\{ -\frac{y}{2t} \right\}. \quad (44)$$

For the case  $\delta = 0$ , the semi-group of  $\text{BESQ}^0$  is equal to

$$Q_t^0(x, \cdot) = \exp \left\{ -\frac{x}{2t} \right\} \varepsilon_0 + \tilde{Q}_t(x, \cdot), \quad (45)$$

where  $\varepsilon_0$  is the Dirac measure in 0 and  $\tilde{Q}_t(x, \cdot)$  has the density

$$q_t^0(x, y) = \frac{1}{2t} \left( \frac{y}{x} \right)^{-\frac{1}{2}} \exp \left\{ -\frac{x+y}{2t} \right\} I_1 \left( \frac{\sqrt{xy}}{t} \right).$$

The transition density for  $\text{BES}^\delta$  is obtained from (43), (44) resp. (45) and is equal to

$$p_t^\delta(x, y) = \frac{1}{t} \left( \frac{y}{x} \right)^\nu y \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} I_\nu \left( \frac{xy}{t} \right),$$

with  $t > 0, x > 0$ , and

$$p_t^\delta(0, y) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu+1)^{-1} y^{2\nu+1} \exp \left\{ -\frac{y^2}{2t} \right\},$$

for  $\delta > 0$ , and the semi-group for  $\text{BES}^0$  is equal to

$$P_t^0(x, \cdot) = \exp \left\{ -\frac{x^2}{2t} \right\} \varepsilon_0 + \tilde{P}_t(x, \cdot),$$

where  $\varepsilon_0$  is the Dirac measure in 0 and  $\tilde{P}_t(x, \cdot)$  has the density

$$p_t^0(x, y) = \frac{x}{t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} I_1 \left( \frac{xy}{t} \right).$$

### A.3 Scaling property.

$\text{BES}^\delta$  processes have the Brownian scaling property, i.e. if  $X$  is a  $\text{BES}_x^\delta$ , then the process  $c^{-1}X_{c^2t}$  is a  $\text{BES}_{x/c}^\delta$  for any  $c > 0$ .  $\text{BESQ}^\delta$  processes have the following scaling property: if  $X$  is a  $\text{BESQ}_x^\delta$ , then the process  $c^{-1}X_{ct}$  is a  $\text{BES}_{x/c}^\delta$ .

### A.4 Additivity property of squared Bessel processes.

An important and well-known property of  $\text{BESQ}^\delta$  processes with  $\delta \geq 0$  is the following additivity property.

**Theorem 5 (Shiga–Watanabe [55])** *For every  $\delta, \delta' \geq 0$  and  $x, x' \geq 0$ :*

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'},$$

where  $Q_x^\delta * Q_{x'}^{\delta'}$  denotes the convolution of  $Q_x^\delta$  and  $Q_{x'}^{\delta'}$ .

For a proof see Shiga–Watanabe [55] or Revuz–Yor[49], p. 410.

## B Time reversal

Consider a transient diffusion  $X$ , living on  $\mathbb{R}_+$ , with  $X_0 = x_0 \geq 0$ . Denote its last exit time of  $a \geq 0$  by  $L_a = \sup\{u \mid X_u = a\}$ , where  $\sup \emptyset = 0$ . For  $a$  fixed,  $L_a$  is finite almost surely, and for  $x_0 < a$ ,  $L_a > 0$  almost surely. We consider the time reversed process  $\tilde{X}$ , where

$$\tilde{X}_t(\omega) \equiv \begin{cases} X_{L_a(\omega)-t}(\omega), & \text{if } 0 < t < L_a(\omega), \\ \partial & \text{if } L_a(\omega) \leq t \text{ or } L_a(\omega) = \infty, \end{cases} \quad (46)$$

where  $\partial$  denotes the ‘cemetery’, and  $\tilde{X}_0(\omega) = X_{L_a(\omega)}(\omega)$ , if  $0 < L_a(\omega) < \infty$ , else  $\tilde{X}_0(\omega) = \partial$ . As a consequence we have the equality

$$\{X_u, u \leq L_a\} = \{\tilde{X}_{T_0-u}, u \leq T_0\}, \quad (47)$$

where we assume  $X$  starting in 0 and  $T_0 \equiv \inf\{u \mid \tilde{X}_u = 0\}$ .

We remark that a diffusion may be reversed at cooptional times, a more general class than last exit times, see Nagasawa [39, 40], Revuz–Yor [49], Chapter VII.4. However, for our purposes here it is reasonable to restrict ourselves to last exit times. In the following, we state a general time reversal result (see Nagasawa [39, 40], Sharpe [53], Gettoor–Sharpe [26], Revuz–Yor [49]). Denote the semi-group of  $X$  by  $(P_t)$ , the potential kernel of  $X$  by  $U$ , and let  $\tilde{\mathcal{F}}_t = \sigma(\tilde{X}_s, s \leq t)$  be the natural filtration of  $\tilde{X}$ .

**Theorem 6** *We assume that there is a probability measure  $\mu$  such that the potential  $\nu = \mu U$  is a Radon measure. Further we assume that there is a second semi-group on  $\mathbb{R}^+$ , denoted by  $(\hat{P}_t)$ , such that  $\hat{P}_t f$  is right-continuous in  $t$  for every continuous function  $f$  with compact support on  $\mathbb{R}^+$  and such that the resolvents  $(U_p)$  and  $(\hat{U}_p)$  are in duality with respect to  $\nu$ , i.e.,*

$$\int U_p f \cdot g \, d\nu = \int f \cdot \hat{U}_p g \, d\nu \quad (48)$$

for every  $p > 0$  and every positive Borel functions  $f$  and  $g$ . Equality (48) can also be written as

$$\langle U_p f, g \rangle_\nu = \langle f, \hat{U}_p g \rangle_\nu.$$

Then under  $P_\mu$ , the process  $\tilde{X}$  is a Markov process with respect to  $(\tilde{\mathcal{F}}_t)$  with transition semi-group  $(\hat{P}_t)$  and we have the duality

$$\langle P_\phi f, g \rangle_\nu = \langle f, \hat{P}_\phi g \rangle_\nu,$$

for any positive Borel function  $\phi$  on  $\mathbb{R}^+$  where  $P_\phi f(x) = \int_0^\infty \phi(t) P_t f(x) dt$ .

We will obtain explicit formulae for time-reversed diffusions via Doob's  $h$ -transform.

## B.1 Doob's $h$ transform.

Consider a one-dimensional diffusion  $X$ , with sample space  $(I^{\partial, \infty}, \mathcal{F}_\infty^c)$ , where  $I \subseteq [-\infty, \infty]$  and  $I^{\partial, \infty} := \{\omega : [0, \infty) \mapsto I \cup \{\partial\}\}$ ,  $\mathcal{F}_\infty^c := \sigma\{\omega(t) | t \geq 0\}$ .

**Definition 5** *A non-negative measurable function  $h : I \mapsto \mathbb{R} \cup \{\infty\}$  is called  $\alpha$ -excessive for  $X$ ,  $\alpha \geq 0$ , if*

- a)  $e^{-\alpha t} E_x(h(X_t)) \leq h(x)$ , for all  $x \in I$ ,  $t \geq 0$ ,
- b)  $e^{-\alpha t} E_x(h(X_t)) \rightarrow h(x)$ , for all  $x \in I$  as  $t \downarrow 0$ .

A 0-excessive function is simply called excessive.

Let  $h$  be an  $\alpha$ -excessive function for a diffusion  $X$ . The life time of a path  $\omega \in I^{\partial, \infty}$  is defined by  $\zeta(\omega) := \inf\{t | \omega_t = \partial\}$ . We construct a new probability measure  $P^h$  by

$$P_x^h |_{\mathcal{F}_t} = e^{-\alpha t} \frac{h(\omega(t))}{h(x)} P_x |_{\mathcal{F}_t}, \quad (49)$$



for  $t < \zeta$  and  $x \in I$ . The process under the new measure  $P^h$  is a regular diffusion and is called *Doob's  $h$ -transform* of  $X$ . As for Doob's  $h$ -transform we refer to Doob [16] and Dellacherie–Meyer [15]; in presenting Doob's  $h$ -transform we followed Borodin–Salminen [6].

As an application let us consider a transient diffusion  $X$  under probability measure  $P$  with scale function  $s$ . Doob's  $h$ -transform of  $X$  with the excessive function  $h \equiv s$ , that is, the process under the new measure  $P^h$ , is a process which reaches 0 almost surely. We have

$$P_x|_{\mathcal{F}_t} = \frac{\left(\frac{1}{s}\right)(X_{t \wedge T_0})}{\left(\frac{1}{s}\right)(x)} P_x^h|_{\mathcal{F}_t}. \quad (50)$$

Note that we obtain

$$\frac{s(X_t)}{s(x)} P_x|_{\mathcal{F}_t} \equiv 1_{(t < T_0)} P_x^h|_{\mathcal{F}_t},$$

and hence,

$$Q_t^h f(x) \equiv \frac{1}{s(x)} E_x [(fs)(X_t)] \equiv \frac{1}{s(x)} P_t(fs)(x), \quad (51)$$

is the semigroup of the process under  $P^h$  killed when it reaches 0. In other words, we have the following time reversal result: Doob's  $h$ -transform of the transient process under  $P$  with  $h = s$  is the process under  $P^h$  killed at  $T_0$ ; this is the process  $\tilde{X}$  in our former notation.

From formula (50) we can obtain explicit formulae of the diffusion processes via Girsanov's theorem. Assume, a process  $(X_t)$  under  $P_x$  has the form

$$X_t = x + \int_0^t \beta(X_u) du + \int_0^t \alpha(X_u) dB_u,$$

then via Girsanov's theorem we obtain with (50) that the process under  $P_x^h$  has the form

$$X_t = x + \int_0^t \left( \beta + \left( \alpha \frac{s'}{s} \right) \right) (X_u) du + \int_0^t \alpha(X_u) d\hat{B}_u, \quad t \leq T_0. \quad (52)$$

## References

- [1] Alili, L. and Gruet, J.C. An explanation of a generalized Bougnol identity. In *Exponential Functionals and Principal Values related to Brownian motion*. Biblioteca de la Ruista Matematica Iberoamericana, 1997.
- [2] Arbib, M.A. Hitting and martingale characterizations of one-dimensional diffusions. *Z. Wahrsch. Verw. Gebiete*, 4:232–247, 1965.

- [3] Ball, C.A. A review of stochastic volatility models with application to option pricing. *Financial Markets, Institutions and Instruments*, 2(5):55–71, 1993.
- [4] Bateman Manuscript Project, A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi. *Tables of Integral Transforms*. Vol. I. McGraw-Hill Book Company, Inc., 1954.
- [5] Bertoin, J. Excursions of a  $BES_0(d)$  and its drift term ( $0 < d < 1$ ). *Probab. Theory Related Fields*, 84(2):231–250, 1990.
- [6] Borodin, A.N. and Salminen, P. *Handbook of Brownian Motion – Facts and Formulae*. Probability and Its Applications. Birkhäuser Verlag Basel Boston Berlin, 1996.
- [7] Breiman, L. First exit times from a square root boundary. *Fifth Berkeley Symposium*, 2(2):9–16, 1967.
- [8] Chen, R. and Scott, L. Pricing interest rate options in a two-factor Cox–Ingersoll–Ross model of the term structure. *The review of financial studies*, 5(4):613–636, 1992.
- [9] Chesney, M., Geman, H., Jeanblanc-Picqué, M., and Yor, M. Some combinations of Asian, Parisian and Barrier options. In Dempster, M.A.H and Pliska, S.R., editors, *Mathematics of Derivative Securities*, Publications of the Newton Institute. Cambridge University Press, 1997.
- [10] Cox, J.C., Ingersoll, J.E. Jr., and Ross, S.A. A theory of the term structure of interest rates. *Econometrica*, 53(2), 1985.
- [11] De Schepper, Goovaerts, M., and Delbaen, F. The Laplace transform of certain annuities with exponential time distribution. *Insurance: Mathematics and Economics*, 11:291–294, 1992.
- [12] Deelstra, G. and Parker, G. A covariance equivalent discretisation of the CIR model. *Proceedings of the 5th AFIR International Colloquium*, 2:731–747, 1995.
- [13] Delbaen, F. Consols in the CIR model. *Mathematical Finance*, 3(2):125–134, 1993.
- [14] Delbaen, F. and Shirakawa, H. Option pricing for constant elasticity of variance model. Preprint, 1996.
- [15] Dellacherie, C., Maisonneuve, B., and Meyer, P.-A. *Processus de Markov (fin); Complements de calcul stochastique*, volume 5 of *Probabilités et potentiel*. Hermann, Paris, 1992.

- [16] Doob, J.L. *Classical Potential Theory and Its Probabilistic Counterpart*. Springer-Verlag New York, 1984.
- [17] Duffie, D. and Singleton, K.J. An econometric model of the term structure of interest rate swap yields. Preprint, Graduate School of Business, Stanford University, 1995.
- [18] Dufresne, D. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand.Actuar.J.*, (1-2):39–79, 1990.
- [19] Eisenbaum, N. Un théorème de Ray–Knight relatif au supremum des temps locaux browniens. *Probab. Theory Related Fields*, 87:79–95, 1990.
- [20] Elworthy, K.D., Li, X.M., and Yor, M. On the tails of the supremum and the quadratic variation of strictly local martingales. In *Séminaire de Probabilités, XXXI*, Volume 1655 of *Lecture Notes in Mathematics*, 113–125. Springer-Verlag Berlin Heidelberg, 1997.
- [21] Elworthy, K.D., Li, X.M., and Yor, M. The importance of strictly local martingales: Applications to radial Ornstein-Uhlenbeck processes. submitted to *Probab. Theory Related Fields*, 1998.
- [22] Frydman, H. Asymptotic inference for the parameters of a discrete-time square-root process. *Mathematical Finance*, 4(2):169–181, 1994.
- [23] Geman, H. and Yor, M. Bessel processes, Asian options, and perpetuities. *Math. Finance*, 3(4):349–375, 1993.
- [24] Genotte, G. and Marsh, T. A. Variations on economic uncertainty and risk premiums on capital assets. *European Economic Review*, 37:1021–1041, 1993.
- [25] Gettoor, R.K. The Brownian escape process. *Ann. Prob.*, 7(5):864–867, 1979.
- [26] Gettoor, R.K. and Sharpe, M.J. Excursions of Brownian motion and Bessel processes. *Z. Wahrsch. Verw. Gebiete*, 47:83–106, 1979.
- [27] Hawkes, J. and Truman, A. Statistics of local time and excursions for the Ornstein–Uhlenbeck process. In *Stochastic Analysis*, volume 167 of *L. M. Soc. Lect. Notes Series*, 91–102. Cambridge University Press, 1991.
- [28] Heston, S. L. A closed-form solution for options with stochastic volatility, with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.
- [29] Horowitz, J. Measure-valued random processes. *Z. Wahrsch. Verw. Gebiete*, 70:213–236, 1985.

- [30] Itô, K. and McKean Jr., H.P. *Diffusion Processes and their Sample Paths*. Springer-Verlag Berlin Heidelberg, reprint of the 1974 edition, 1996.
- [31] Jeulin, T. *Semi-Martingales et Grossissement d'une Filtration*. LNM 833. Springer-Verlag Berlin Heidelberg, 1980.
- [32] Kent, J. Some probabilistic properties of Bessel functions. *Ann. Prob.*, 6(5):760–770, 1978.
- [33] Knight, F.B. *Essentials of Brownian motion and diffusion*. Math. Surv. 18. Amer. Math. Soc. Providence RI, 1981.
- [34] Lamperti, J. Semi-stable Markov processes I. *Z. Wahrsch. Verw. Gebiete*, 22:205–225, 1972.
- [35] Lebedev, N.N. *Special Functions and their Applications*. Dover Publications, New York, 1972. Translated and edited by Richard A. Silverman.
- [36] Leblanc, B. Une approche unifié pour une forme exacte du prix d'une option dans les différents modèles à volatilité stochastique. *Stochastics Stochastics Rep.*, 57(1-2):1–35, 1996.
- [37] Leblanc, B. *Modélisations de la Volatilité d'un Actif Financier et Applications*. PhD thesis, Université de Paris VII, 1997.
- [38] Millar, P.W. Random times and decomposition theorems. *Proc. Symp. Pure Mathematics*, 31:91–103, 1977.
- [39] Nagasawa, M. Time reversion of Markov processes. *Nagoya Math. J.*, 24:177–204, 1964.
- [40] Nagasawa, M. *Schrödinger Equations and Diffusion Theory*, volume 86 of *Monographs in Mathematics*. Birkhäuser Verlag Basel Boston Berlin, 1993.
- [41] Novikov, A.A. A martingale approach to first passage problems and a new condition for Wald's identity. In *Lecture Notes in Control and Information Sci.*, Volume 36 of *Stochastic differential systems (Visegrád, 1980)*, 146–156. Springer-Verlag Berlin New York, 1981.
- [42] Novikov, A.A. A martingale approach in problems on first crossing time of nonlinear boundaries. *Proc. of the Steklov Institute of Mathematics*, 4:141–163, 1983.
- [43] Novikov, A.A. Limit theorems for the first passage time of autoregression process over a level. *Proc. of the Steklov Institute of Mathematics*, 4:169–186, 1994.
- [44] Oberhettinger, F. and Badii, L. *Tables of Laplace transforms*. Springer-Verlag, Berlin, 1973.

- [45] Pitman, J. and Yor, M. Bessel processes and infinitely divisible laws. In *Stochastic Integrals*, Volume 851 of *Lecture Notes in Mathematics*, 285–370. Springer-Verlag Berlin Heidelberg, 1980.
- [46] Pitman, J. and Yor, M. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59:425–457, 1982.
- [47] Pitman, J. and Yor, M. Sur une décomposition des ponts de Bessel. In *Functional Analysis in Markov Processes*, Volume 923 of *Lecture Notes in Mathematics*, 276–285. Springer-Verlag Berlin Heidelberg, 1982.
- [48] Pitman, J. and Yor, M. Laplace transforms related to excursions of a one-dimensional diffusion. *Bernoulli*, 5(2):249–255, 1999.
- [49] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*. Springer-Verlag Berlin Heidelberg, 1991.
- [50] Ricciardi, L.M. and Sato, S. First-passage time density and moments of the Ornstein–Uhlenbeck process. *J. Appl. Prob.*, 25:43–57, 1988.
- [51] Rogers, L.C.G. Characterizing all diffusions with the  $2M - X$  property. *Ann. Probab.*, 9(4):561–572, 1981.
- [52] Salminen, P. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *Adv. in Appl. Probab.*, 20(2):411–426, 1988.
- [53] Sharpe, M.J. Some transformations of diffusions by time reversal. *Ann. Prob.*, 8(6):1157–1162, 1980.
- [54] Shepp, L. First passage problem for the Wiener process. *Ann. Math. Statist.*, 38:1912–1914, 1967.
- [55] Shiga, T. and Watanabe, S. Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 27:37–46, 1973.
- [56] Siegert, A.J.F. On the first passage time probability problem. *Phys. Rev.*, 81(4):617–623, 1951.
- [57] Szatzschneider, W. Bessel processes and pricing interest rate derivative securities in generalized CIR model. Universidad Anaáhuac, Mexico City.
- [58] Truman, A. and Williams, D. A generalised arc-sine law and Nelson’s stochastic mechanics of one-dimensional time-homogeneous diffusions. In: *Diffusion processes and related problems in analysis*. *Progr. Probab.*, 22:117–135, 1990.
- [59] Widder, D. V. *The heat equation*. Academic Press, Inc., 1975.

- [60] Williams, D. Path decomposition and continuity of local time for one-dimensional diffusions, I. *Proc. London Math. Soc. (3)*, 28:738–768, 1973.
- [61] Yor, M. On square-root boundaries for Bessel processes and pole-seeking Brownian motion. In *Stochastic Analysis and Applications*, Volume 1095 of *Lecture Notes in Mathematics*, 100–107. Springer-Verlag Berlin Heidelberg, 1984.
- [62] Yor, M. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.*, 24(3):509–531, 1992.
- [63] Yor, M. *Some Aspects of Brownian Motion, Part I: Some special functionals*. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag Basel, 1992.
- [64] Yor, M. Sur certaines fonctionnelles exponentielles du mouvement Brownien réel. *J. Appl. Prob.*, 29:202–208, 1992.
- [65] Yor, M. *Local Times and Excursions for Brownian Motion*. Volume 1. 1995 of *Lecciones en Matemáticas*. Universidad Central de Venezuela, 1995.
- [66] Yor, M. *Some Aspects of Brownian Motion, Part II: Some recent Martingale Problems*. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag Basel, 1997.