# Pricing and Risk Analysis of correlation Products: Evidence of Synthetic CDO Swaps.

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# Abstract

As popular vehicles for trading a portfolio of credit risks, we focus on a Synthetic Collateralized Debt Obligation swaps (Synthetic CDOs), in terms of pricing and risk analysis. Our purpose is not to create a new concept in these *stylised facts* of correlation products. Instead, we attempt to assess the key idea behind the standard credit derivatives pricing model in order to fully capture the essential of the risk of a synthetic CDO swaps**.**

To this end, we provide a step by step description of the one factor Gaussian Copula model which is said to overcome computation costs inherent to the use of Monte Carlo simulation in the standard Gaussian copula model. This thesis also presents the *double-t* distribution suggested by Hull and White (2004) as an extension of the one factor Gaussian copula where they used a multi factor framework. For practical purpose, we use Microsoft Excel to calculate a synthetic CDO tranche price based on the computation of a homogenous portfolio of credit defaults under the one factor Gaussian copula model. We compared our empirical results in terms of prices relative to our homogenous assumptions with the market quotes. We recognized that even if the CDO pricing theoretical side in terms of relationship between the default correlation risk and tranches prices is satisfied, our model prices do not match the market quotes.

The thesis then goes on to present a way to assess the demanding credit risk analysis in light of such appealing issue. We also introduce other problems that we would like to understand better such as the implied and base correlations. We highlight the intuition behind them in terms of pricing and risk analysis. Finally the recent trouble of Bears Stearns funds' is assessed.


Key words: Survival function, joint distribution, loss distribution, Gaussian copula, Factor copula, probability bucketing, base correlation, implied correlation.

## CONTENTS

# I- INTRODUCTION

Credit derivatives are financial instruments which allow banks, and other financial institutions to efficiently transfer and manage credit risks. Furthermore Credit derivatives market participants can reduce the cost of regulatory capital.

Credit default swaps (CDS) as the key component of the credit derivatives market in terms of volume, has seen a substantial growth with around US $181 billion notional outstanding in December 2006 as highlighted by Bank of International Settlements (BIS). However, other instruments originally based on the CDS have seen subsequent growth as well, among them Basket Default Swap, Collateralized Debt Obligations (CDOs) and single tranche synthetic CDOs (STCDO).

In order to understand the credit market think of a holder of bond who is rewarded for exposure to some risk. Imagine that the issuer of the bond defaults. In such a case, the investor (holder of the security) loses all her/his investment. At this point the investor could enter into a contract to buy protection against such a default. To this end, our investor agrees to pay a premium periodically to the seller of the protection until the maturity of the transaction or until a default occurs. On the other hand, the seller of the protection agrees to give the par bond if the issuer of the bond defaults. Thus the main risk of default is transferred effectively. Basically this is called a credit default swap (CDS).

The protection seller could prefer a basket of bonds for diversification purpose, such that if one bond defaults, he/she could not suffer huge loss. This is partly a reason for Banks, as protection sellers to invest in a portfolio of credit derivatives such that their profit is not strongly affected should one credit instrument defaults in their portfolio. For risk management purposes among others motivations, banks divide their portfolio of debt securities such as bonds or loans into several tranches (equity, mezzanine, senior) with different risk exposure as described in Verschuere (2005). These tranches are called Collateralized Debt Obligations or CDOs tranches for simplicity. The originators of these tranches sell them to some investors with different prices relative to the level of risk they bear. Hence the equity tranche, as the most risky, is the most expensive tranche. Note that Choudry (2002) reported this instrument as a *true sale* since the underlying instruments are bonds or loans. The trade is effective until the whole set of the CDO tranches are sold. Generally the issuer of such instruments retains the equity tranche. Synthetic collateralised debt obligations or synthetic CDOs are in contrast backed by a reference portfolio of credit default swaps (CDS) or other instruments such as Total Return Swap (TRS) instead of cash assets like bonds.

Defining the price of each tranche for which an investor is ready to support the risk of default of the firms in the reference portfolio of cash and/or synthetic CDOs is the key challenge in credit derivatives markets. The payoff of such instruments is driven by the default dependency of pair wise firms in the reference portfolio as mentioned in Schoenbucher and Schubert (2001). Note that pricing credit derivatives instruments based on the default correlation is not an easy task. For instance Verschuere (2005) recognized the difficulty of such an exercise. The complexity in pricing issues leads necessarily to a difficult risk analysis, thereby demanding risk management strategies. Cited by Financial Times (2005), Alan Greenspan, a former chairman of the US Federal Reserve Bank argued that: *"….Understanding the credit risk profile of CDO tranches poses challenges even to the most sophisticated participants"*. Hence capturing the risk embedded in the CDO tranches can be seen as the key to a successful credit derivatives management. This is not always easily the case considering the recent trouble in the credit derivatives market. Recall that this risk is strongly related to the firms' default correlations. In fact default dependency infers in hedging strategy of the traded structures. The main challenge therefore in valuing correlation products remains the specification of the joint distribution of the default arrival time of firms in play given their marginal distributions as discussed in Chen and Glasserman (2006).

In this vein, from structural models, pioneered by Merton (1974) and further developed by Zhou (2001), to intensity-based models (see Lando, 1998, Duffie and Singleton, 1999) various models have been developed to capture the spirit of default correlation. Balakrishna (2007), recently recognized the lack of a straightforward solution to perform a perfect correlation structure. However, the most accepted pricing model in the industry is the so called Normal (Gaussian) Copula as discussed in Li (2000) and further developed by amongst others, Gregory and Laurent (2003). Note that the computation of the Gaussian copula model requires the Monte Carlo simulation framework which is said to be time consuming. This technique is also used as a core instrument in Credit Metrics (see Gupton, Finger and Bahia, 1997) and allows one to specify a joint distribution by combining marginal default arrival times of firms in the portfolio and their pair wise correlation as stressed in Daglish and Li (2005).

Other copula based models have been developed such as Student t-copula (Mashal and Naldi (2001) or Clayton copula (Rogge and Schoenbucher (2003)) but the Gaussian copula remains the widely used method. However, Gregory and Laurent (2003, 2004), Finger (2004) amongst others noted that the Gaussian copula model bears serious drawbacks, first of all in terms of correlation skew implied from the market quotes where the correlation is said to be flat and secondly in terms of computational cost involved in the use of Monte Carlo simulation. Furthermore the normal (Gaussian) copula model is said to be limited to pricing non-standard multiname credit derivatives (for example bespoke tranches) as remarked by Hull and White (2004). This is where the one factor approach comes in based on a common factor underlying the specification of the default correlation.

Hence alternative ways have been investigated. Amongst others Baxter (2006) used Joshi and Stacey (2005) statement to perform a model base on default intensities in order to resolve the problem of the correlation skew while, based on Gregory and Laurent's (2003) earlier work, Hull & White (2004) developed two appealing approaches which are supposed to ease the CDO tranches and nth-to-default swap pricing. As we noted above, due to a certain doubt about the standard model to perfectly modelling correlations products, the probability of incorrectly pricing such instruments is high. At this point, David Li, the Gaussian copula model pioneer, warned market participants against the potential trouble that investors who *believe without limits* in the model's outcome will face as reported by the Wall Street Journal (2005). In such an atmosphere it is obvious for market participants to make some misspecification in risk analysis and trading strategies of correlation dependent products. An obvious consequence is the loss of huge amounts of capital. From this, Elizalde (2006) discussed the fact that some investors in the CDOs market are said not to fully understand the risks underlying the trade of these products.

The remaining part of the thesis is structured as follows: the second chapter provides the necessary mathematic background to capture the key concepts of copula and its properties as discussed in Andersen (2006).

Chapter 3 presents the description of the default correlation modelling. Here we introduce the Gaussian copula model. The fourth chapter presents the widely used one factor Gaussian copula that has become the market standard for CDOs tranches pricing. Our goal here is to make the model easy to understand for those who actively participate in the credit derivatives markets, with low mathematic level. We then fully describe each component of the model.

A detailed synthetic CDO tranche pricing process will be described in Chapter 5, based on the one factor Gaussian copula such that the cost of the well recognized *expensive* Monte Carlo Simulation used to compute the Gaussian copula is avoided. In this chapter, before tackling a description of the multi factor copula model as suggested in Hull and White (2004) as an extension of the standard model, we will define a synthetic CDO in terms of structure and mechanics. We will also compute the one factor Gaussian copula using Microsoft Excel.

Chapter 6 will assess different risks embedded in the synthetic CDOs tranche pricing model as discussed in Gibson (2004). Furthermore, given the non-uniqueness of default correlation evaluation methods, see Daglish and Li (2005). The key target of this section, thanks to McGinty et al (2004) and Kakodkar et al (2006), is to capture the impact of the implied and base correlations in terms of pricing and risk analysis. Here we clearly stress the default correlation as one of the price driver in modelling portfolio losses. The thesis then goes on to explain the extent to which the business cycle can impact the exposure of a synthetic CDO tranche holder. This is where we assess the recent trouble of the Bears Stearns hedge funds. The last chapter concludes with some remarks.

# II. PRELIMINARIES.

Since the 1990's Credit Derivatives market is growing, millions of pounds are reported as investments banks, insurance companies and hedge funds annual benefits due to "successful" management of their credit risk portfolio. On the other hand no one care about what these companies loose every year in their transactions due to error or misspecification of the risks in play. Note that Credit default swaps (CDS) are the most traded instruments in the credit market. They also constitute the building blocks for other credit products. Recent developments in the industry have seen the growth of so called correlation products such as Collateralized Debt Obligation (CDOs) and synthetic CDOs. The latter are different from the CDS in that they involve default correlation features, i.e. the principal price driver in this case is the pair wise default correlation of firms in the portfolio. Understanding the intuition behind CDOs' pricing models leads to an efficient risk analysis, thereby to improving the risk management. Unfortunately, Elizalde (2006) mentioned that a non negligible percentage of market participants do not capture the mechanism of default correlation measure and as a result lose huge amounts of Capital that they never talk about.

In this chapter, we describe the inputs for a good understanding of the key model used in the credit derivatives market. Hence the first section, based on Andersen (2006) and Galiani (2003), set up the Poisson or Cox process as the building block theory underpinning the entire methodology used in correlation instruments valuation. Secondly, thanks to the outcome of the Poisson (Cox) process, we will define the survival function from which we derive the probability of default arrival time as an important input in pricing credit risk products. The third point in this chapter leads necessarily to the probability distribution function of the default time. Next we use the above information to deduce, in section four, the probability of default in a small time interval ***t + dt***, given the fact that the firm has survived during a certain life time $[0,t]$ that is the instantaneous default probability or the **probability density function**. Note that the probability density function is used to produce a credit curve by a bootstrapping procedure that involves the credit spread. Credit curve is the main input in measuring the default correlation for a pool of credit risk instruments as we will see later.

This chapter will end with a sixth section where we explain the “loss-given-default” process which will help in deriving the loss distribution as we shall see later in this thesis.

## II-1 Poisson and Cox processes.

The probability distribution of the marginal loss given default in the reference portfolio is of crucial importance in measuring the risk in credit derivatives environment. It led to the construction of the credit curve which displays the instantaneous default probabilities of each reference entity in the portfolio as mentioned in Galiani (2003).

Two main types of models have been developed to capture the to default time distribution: Structural models pioneered by Merton (1974) and Reduced-form models. The latter is the most used in the credit derivative trading area as recognized by Andersen et al (2003).

In this section, we focus on some important concepts which led to a better understanding of the spirit of the reduced-form models that is Poisson and Cox processes.

To this end assuming a fixed time period, a certain number of events may occur during that specified length of time. As mentioned in the Journal of Archive for History of Exact Sciences (1984), Simeon-Denis Poisson in 1838 figured out some random variables N that specify the number of discrete default events which may occur during that specified time of period. He found that the probability that there are exactly k default events during the considered length of time is: $f(k;\lambda) = \dfrac{e^{-\lambda}\lambda^{k}}{k!}$

Here $\lambda$ is the expected number of events which depends on time. Assuming that $N_t$ corresponds to the number of default events before time t, the probability that exactly k events occur before that time t is: $\Pr(N_t = k) = f(k;\lambda t) = \dfrac{e^{-\lambda t}(\lambda t)^{k}}{k!}$.

At this point, the probability that there is no default event before time t, hence k = 0 can be written as:

$\Pr(N_t = 0) = e^{-\lambda t}\dfrac{(\lambda t)^{0}}{0!} = e^{-\lambda t}$. Note in terms of expectation, the number of default events can be written as $E(N(t)) = \lambda t$.

Assuming now that $\tau$ represents the event's arrival time. Thus probability that there is no default before time t can be explain as the probability that $\tau > t$. As a mathematical expression we can write: $\Pr(\tau > t) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$.

Accordingly $\tau \le t$ represents the fact that a default event occurs before time t. At this point, we can write $1_{\{\tau \le t\}}$ as the function of default event.

From this, the probability of default event can be written as: $\Pr(\tau \le t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} = e^{-\lambda t}$ .

where k represents the number of default events**.** Note that the equation above can be written as $\Pr(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} = e^{-\lambda t}$ where $N_t$ is also an *integer process* which takes only increasing values 0,1,2….*n*. As such, the increasing process $N_t$ is the **Poisson process**. The parameter $\lambda$ is also defined as the intensity of the poisson process, see Andersen (2006) .

We can summarise this key process as follow:

1- $\Pr(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ the probability of k default events.

2- $E(N(t)) = \lambda t$ the expected number of default events.

3- $\Pr(\tau > t) = e^{-\lambda t}$ the probability of no default event.

An important result which can be seen as the hazard rate is the instantaneous default event that occurs after a certain lifetime t.

4- $\Pr(\tau \in [t, t + dt]) = \lambda e^{-\lambda t} dt$ .

Indeed, the latter result above can be explained as the probability that a default event occurs in a small length of time after a certain lifetime t. Note also that the poisson process has some Markovian feature such that there is no memory beyond the present as stated Wilmott (2002).

Since the intensity of the poisson process as stated above is time dependent, we can replace $\lambda t$ by $\int_0^t \lambda(u)d(u)$ to take into account the integer process involved in the poisson process. This leads to *key results* as below:

$$\Pr(N_t = k) = \exp\left(-\int_0^t \lambda(u)du\right)\frac{\left(\int_0^t \lambda(u)du\right)^k}{k!} \quad \textbf{(1)}$$

$$E(N(t)) = \int_0^t \lambda(u)du \,. \quad \textbf{(2)}$$

$$\Pr(\tau > t) = \exp\left(-\int_0^t \lambda(u)du\right). \quad \textbf{(3)}$$

$$\Pr(\tau \in [t, t+dt]) = \lambda(t)\exp\left(\int_0^t \lambda(u)du\right)dt \,. \quad \textbf{(4)}$$

From this, we assume that the intensity of the poisson process $\lambda$ is stochastic, dependent on the time such that given a certain interval of time $[0,t]$ the first default event conditional on the time can be written as:

$$\Pr(\tau 1 \leq t / \{\lambda(u), 0 \leq u \leq t\}) = 1 - \exp\left(-\int_0^t \lambda(u)du\right) \quad \textbf{(5)}$$

Where $\tau 1$ represents the first default event within the period of time $[0,t]$ . We then have a conditional probability of default event. At this point, evaluating default event is not easy because the integration process has to take into account the previous default probability. This fact is evident when we remember that the poisson process, as a Markovian process has no memory. This is where as reported the Wikipedia website Sir David Cox comes in by forming an expectation over all the paths that default events follow as pointed out by Andersen (2006) such that:

$$\Pr(\tau \leq t) = 1 - E(\exp\left(-\int_0^t \lambda(u)du\right)). \quad \textbf{(6)}$$

The equation above represents the **Cox process,** a generalization of Poisson process.

### II-2 Time until default or Survival function.

In this section, thanks to the Poisson (Cox) process, we focus on the so called survival function in order to later define accordingly the default distribution function of single name credit derivatives.

Note that modelling the default time is the key to figure out an essential outcome in resolving the problem of co-dependency of default time in collateralised debt obligations (CDOs) pricing as we shall see later in this thesis.

In fact, modelling multiname credit derivatives products such as synthetic CDOs requires the knowledge of the default intensity curve (or credit curve), which represents the instantaneous default probability for each entity in the portfolio of credit derivatives.

To this end we assume that the default time corresponds to the first jump (first default event) of the Poisson process $N_t$ with default intensity $\lambda$. Thus $\tau = \inf\{t > 0 : N(t) = 1\}$.

Remember the equation **(3),** that is $\Pr(\tau > t) = \exp\left(-\int_0^t \lambda(u)du\right)$**.** We can explain this formula as the fact that there is no default event before the considered length of time *t.* Then the first default may occur obliviously after that length of time. We call this period of time, the time until default or the survival time such that $\tau > t$.

We note in general $S(t) = \Pr(\tau > t) = \exp\left(-\int_0^t \lambda(u)du\right)$ as the **survival function**. We can generalise this expression in terms of expectation according to the Cox process as:

$$S(t) = \Pr(\tau > t) = E(\exp\left(-\int_0^t \lambda(u)du\right)) \quad \textbf{(7).}$$

**II-3 Default time distribution function.**

Once we have defined the time until default of a single firm, we can easily deduce the default time distribution which is of crucial importance in pricing a pool of credit derivatives or correlation products in that it gives us the probability of default within a specific period of time for single firm.

Note the indicator function of default time is $1_{\{\tau \le t\}}$ and as stated above the probability of default during the length of time $[0,t]$ can be written as $\Pr(\tau \le t) = 1 - \Pr(\tau > t) = 1 - S(t)$.

We therefore understand that the default time distribution function can be written as:

$$F(t) = 1 - S(t) = 1 - \exp(-\int_0^t \lambda(u)du \quad \textbf{(8)}.$$

Hence the distribution function can be derived from the survival function. This result is very important for deriving the term structure of credit risk.

**II-4 Probability density function**

We have so far commented on different components, useful for correlations products pricing. We will use them in this section to set up key feature of multiname credit derivatives products modelling. That is the estimation of the instantaneous default probabilities. Here again Poisson (Cox) process is helpful in that the equation **(4)**, $\Pr(\tau \in [t, t+dt]) = \lambda(t) e^{-\int_0^t \lambda(u)du} dt$ can be explained as the probability that default occurs in a small interval of time *(t, t + dt)* after a certain survival time *t (i.e.*no default occurs between time 0 and time *t)*.

This expression can be used effectively to estimate the instantaneous default probabilities for each reference entity that we need in pricing our portfolio of credit derivatives. Note that the so called instantaneous default probability is the unconditional default probability between time *t* and *t + dt.* We called ***f (t)***[1] the probability density function as:

***f(t)***= $\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)$ **(9)**

1.Given that $\lambda$ (t) (i.e. the default intensity) = h(t) (i.e. the hazard rate) , see Galiani (2003) for more details,

$$\lambda(t) = \lim_{\Delta t \to 0} \Pr(t < \tau \le t + \Delta t / \tau > t). = \lim_{\Delta t \to 0} \frac{\Pr(t < \tau \le t + \Delta t / \tau > t)}{\Pr(\tau > t)}$$

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f(u)du}{\int_t^{\infty} f(u)du} = \frac{f(t)}{S(t)} = \frac{f(t)}{\exp\left(-\int_0^t \lambda(u)du\right)}$$

$$f(t) = \lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)$$

## II-5 Default intensity curve

Credit default swaps (CDS) are the most traded instruments in the credit derivatives market. This contract transfers credit risk from one party to another. The buyer of protection agrees to pay periodically a fixed amount (premium) during the contract lifetime, to the protection seller as long as default has not occurred. On the other hand, the protection seller promises to pay a contingent claim, typically the loss given default of the nominal value, that is: $l = 1 - R$, should the issuer of the bond default during the lifetime of the contract. Note that $R$ is he recovery rate.

In case of default, the premium payment ceases but the protection buyer will pay an accrued interest to its counterpart.

The fair price of the CDS is then a percentage of the principal such that the expected present values of both the premium leg and the contingent payment (default leg) are equal. For further understanding of the CDS pricing process one can refer to Hull (2000).

For now note that the pricing of synthetic CDOs, as the topic of this thesis, requires a term structure of default intensity. In practice, one can derive the intensity curves using default swap (CDS) spreads from the markets by using a *"Bootstrapping"* process. For more details on this technique see Galiani (2003), Kakodkar et al (2006).

Given the market quotes for CDS with a constant recovery rate and an assumption of a constant default intensity, Andersen (2006) assess a *good approximation* of default intensity as:

$$\lambda = \frac{CDSspread}{1 - R} \qquad \textbf{(10)}.$$

Where $\lambda$ is the default intensity while 1 – R represents the loss-given-default. Note that the credit curve of individual firm that results is of crucial importance when considering the valuation of a pool of credit risks.

**II-6 Loss Distribution**

In the previous section we have seen that pricing a portfolio of credit risks, evolves a term structure of default for individual firm. We also illustrated the fact that, the buyer of the protection receives, from the protection seller; the par loss should a default triggered.

In this section thanks to Andersen (2006) we explain the mathematical notation behind this idea. Assume $l_i$ be the marginal loss in the portfolio, that is $l_i = 1 - R$.

We can therefore write the individual loss-given-default as: $l_i 1_{\tau \le T}$ where $1_{\tau \le T}$ indicates the default arrival function. Form this, the portfolio loss-given-default $L(T)$ can therefore be written as the sum of individual loss-given-default.

Hence $$L(T) = \sum_{i=1}^{N} l_i 1_{\tau_i \le T} \qquad \textbf{(11).}$$

From this we can compute the portfolio expected loss-given-default as:

$$E(L(T)) = \sum_{i=1}^{N} l_i E\left(1_{\tau_i \le T}\right) = \sum_{i=1}^{N} l_i F_i(t) \qquad \textbf{(12).}$$

Where $F(t)$ is the default distribution function. Thus the portfolio expected loss-given default distribution can be written as:

$$E(L(T)) = \sum_{i=1}^{N} l_i \left(1 - \exp\left(-\int_0^t \lambda(u) du\right)\right) dt \,. \qquad \textbf{(13).}$$

**III- MODELLING DEFAULT CORRELATION.**

Credit derivatives market's participants most of the time use standard techniques such as portfolio diversification to protect themselves against systematic risk (i.e. the risk relative to the market). However, the development of new and complex structured credit products has increased the uncertainty. From this point, playing on credit risk on its own right is not an easy task; particularly a game which involves a portfolio of firms seems to be a *black box*. Galiani et al (2006) recognized that the measurement of the default correlation remains a difficult *parameter to determine.* At this point, Li (2000) introduced the Gaussian Copula, further developed by Laurent et al (2003), Andersen et al (2003), Hull and White (2004), that captures such a crucial input in credit risk analysis and Risk management strategies

The Gaussian copula model is widely used in the credit market today. In this part of the thesis we first define the default correlation and the joint default probability as main inputs for the Copula function. The second section investigates the Copula function as the building block to modelling default correlation given a pool of credit risks. The last section deals with the standard model used in the credit derivatives market: the normal (Gaussian) Copula.

**III-1 Default Correlation and Joint Default Probability.**

As we said above, pricing credit derivatives given a portfolio of risky assets involves taking into account the pair wise correlation of default. At this point we can state that the portfolio default intensity depends on the correlation between firms in play. Intuitively, we can say that the higher the co-dependence of firms, the higher the correlation between them therefore the higher a dominos effect of default within the whole set of the portfolio (i.e. the tendency for firms to default together). Then measuring the co-dependency of firms can help to accurately valuing default correlation based products. We can use the usual definition of correlation to set up the joint default correlation (i.e. a pair wise default correlation in this case) as in Li (2000).

Assume $P_A$ and $P_B$ , are the default event probability of firm A and firm B respectively. Note that these event depend on the interval of time $[0,T]$ (…..it is crucial to subordinate the default event to time because among other reasons, defining payoff should default occurs must be a discounted value at time of default). Thus, the default correlation can be written as :

$$\rho_{AB} = \frac{\text{cov}(P_A P_B)}{\sqrt{Var(P_A)Var(P_B)}} = \frac{P_{AB} - P_A P_B}{\sqrt{P_A^2 - P_A P_B}\cdot\sqrt{P_B^2 - P_B P_A}} \quad \textbf{(14)}$$

Assuming $P_A^2 = P_A$ and $P_B^2 = P_B$ , are known. It appears clearly that the default correlation $\rho_{AB}$ is linearly related to the parameter $P_{AB}$ .

Note the parameter $P_{AB}$ represents the joint probability of default of the firm A and B. Remember the individual default probability of the firm A and B can be written as:

$P_A = 1_{\tau_A \le T} = \Pr(\tau_A \le T)$ and $P_B = 1_{\tau_B \le T} = \Pr(\tau_B \le T)$ .

Hence, the joint default probability can be written as:

$$P_{AB} = \Pr(\tau_A \leq T, \tau_B \leq T) \quad \textbf{(15).}$$

The process above is straightforward in determining the joint default probability. However, the linear correlation which involves has attracted criticism. For instance Li (2000) defined the linear correlation as discrete and leads to wasting important information in need (i.e. the information within the entire interval of time considered rather than a punctual one). This is where Copula comes in, pioneered by Li (2000) in the area of finance.

**III-2 Copula function**

Individual firms bear sufficient information that is characterised by their credit curves, it is very important to keep that necessary information in mind when determining their joint distribution.

As we said the discrete default correlation (i.e. linear correlation) does not match that statement. Remember individual default events are random variables and are normally distributed in a continuous framework over a specified interval of time, such that there is no room for a discrete statement.

At this point, we need a distribution function which can bind all the marginal distribution functions to form a joint distribution function in a continuous framework.

Galiani et al (2006) stressed the difficulty that emerges in the formulation of the joint distribution because the number of joint distributions increases exponentially as a function of the number of firms considered. This where the copula function comes in. It is said to meet the characteristics defined above such that it can be defined given the Sklar (1959) theorem (we define it later in this section), as a multivariate joint distribution function which linked a pool of default curves with a unique multidimensional default curve.

For the purpose of mathematical construction, recall:

$1_{\tau \leq T}$ : is the indicator function of default.

$F_i(t)$: is the marginal default probability distribution function, remember that default time $\tau_i$ is a uniformly distributed random variable where *i = 1, 2, ………….n* representing the number of firms. At this point we can write the portfolio of default distribution functions as:

$$F_{portfolio}(t) = \{F_1(t), F_2(t), ..........., F_N(t)\} = \Pr(\tau_1 < t, \tau_2 < t, ........., \tau_N < t) \quad \textbf{(16)}$$

From Sklar theorem, these marginal distribution functions can be bound into a unique multidimensional default probability distribution, which is called Copula and written as:

$$C[F_1(t), F_2(t), \ldots\ldots\ldots, F_N(t)] = \Pr(\tau_1 < t, \tau_2 < t, \ldots\ldots\ldots, \tau < t) \quad \textbf{(17)}$$

Copula represents the joint distribution. Given the time ***t,*** think of the probability distribution function $F(t)$ as the mechanism from which, the default probability $\Pr(\tau < t)$ is deduced. From this, an important finding of Sklar theorem allow us to consider a backward move such that given the default probability $\Pr(\tau < t)$ we can compute the time $t$ by inverting the mechanism $F(t)$ via an inverted distribution function $F^{-1}$, see Galiani et al (2006).

Hence we have: $t_i = F_i^{-1} F_i(t)$ **(18)**.

This expression can be achieved by using the **Normsinv** function in Microsoft Excel.

In light of the inversion mechanism above, we can write our Copula as follow:

$$C[F_1(t), F_2(t), \ldots\ldots\ldots, F_N(t)] = \Pr(\tau_1 < F^{-1}F_1(t), \tau_2 < F^{-1}F_2(t), \ldots, \tau_N < F^{-1}F_N(t))$$

$$C[F_1(t), F_2(t), \ldots\ldots\ldots, F_N(t)] = \mathbf{H}\{F^{-1}F_1(t), F^{-1}F_2(t), \ldots\ldots, F^{-1}F_N(t)\} \quad \textbf{(19)}$$

Where $\mathbf{H}\{.,.\}$ characterized the joint distribution function of the whole set of margins represented by their respective inverted distribution functions. Another important result from the Sklar's theorem is that marginal distribution functions (i.e. $F_i(t)$ ) can be separated from their pair wise default correlation $R$. Such that we can have the presentation below:

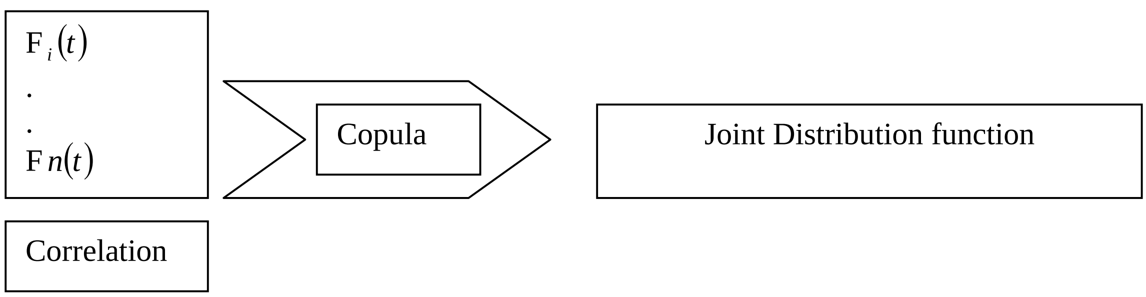


**Figure1: the Copula function structure**

### III-3 Gaussian (normal) Copula.

So far we have described the process of default event, given a certain interval of time period, as a default distribution function $F_i(t)$, that is the probability that a default occurs during a certain the period of time say $[0,T]$ where $t \leq T$ . We have also seen how these marginal distribution functions are assembled to form a joint distribution via the copula function.

Now we define that distribution function in other words. In fact, the probability of the event that the random variable $\tau$ is less or equal to $t$, that is $\Pr(\tau \leq t)$ can be defined as the cumulative distribution function (cdf) say $\Phi_i(t)$.

This is exactly the cumulative distribution function of a standard normal distribution say $\Phi\left(\frac{t-\mu}{\sigma}\right)$ where the mean $\mu = 0$ and the standard deviation $\sigma = 1$.

Now, as we defined above, assume the inverted distribution is $\Phi^{-1}$. We can write the joint distribution function as:

$$\Phi_N\left(\Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), .........,\Phi^{-1}(F_N(t))\right) \quad \textbf{(20)}$$

Where $\Phi_N$ is the cumulative distribution function for ***N*** firms.

Note that the normal distribution also called Gaussian distribution is due to a banker Li (2000). He linked the copula function with the standard normal distribution to the point that the Gaussian or normal copula function has been qualified as an innovation in the credit derivatives market as highlighted by Darrel Duffie in the Wall Street Journal (2005).

From this we arrive now to specify the joint distribution in the Gaussian Copula framework. Remember key characteristic of Copula are such that the whole set of marginal distributions $F_i(t)$ can be separated from the pair-wise default correlation ***R.*** As a result, for $N$ firms we have a $N \times N$ default correlation matrix ***R***.

At this point the **Gaussian Copula** can be written as:

$$C_G[F_1(t), F_2(t), ..........,F_N(t)] = \Pr(\tau_1 < t, \tau_2 < t, .., \tau_N < t)$$

For $t = \Phi^{-1}(F(t))$

$$\Pr(\tau_1 < t, \tau_2 < t, .., \tau_N < t) = \Phi_N\left(\Phi^{-1}(F_1(t)), \Phi^{-1}(F_2(t)), .........,\Phi^{-1}(F_N(t)), R\right) \quad \textbf{(21)}$$

Where $\Phi_N$ is the cumulative distribution function for the portfolio of ***N*** firms with correlation matrix ***R***. Note that, the default correlation has been introduced. We also see that the default correlation is in fact of an important input when computing the cumulative default time. At this point we can perform the evaluation of the default times $\tau_i$ where $i = 1,2,.........., N$ indicate the number of firms. To this end, a numerical approach to compute the Gaussian Copula model is the Monte Carlo simulation. Key fact in this case is the use of Cholesky decomposition such that the correlation matrix R is the product of a $N \times N$ lower triangular matrix A and its transpose $A^T$. Such that the correlation matrix is $R = AA^T$. One can find detailed literature in Li (2000).

Galiani et al (2006), Laurent et al (2003) amongst others stressed the high number of estimates in play thereby the time consuming features of the method.

Fortunately there exist alternative procedures that capture the correlation between credits default times. Many researches have pointed out the introduction of factor approach in Gaussian Copula, as beneficial. For instance, Laurent et al (2003) used Fast Fourier Transform (FFT) to speed up the computation of the one factor Gaussian copula model. While Andersen et al (2003) focused on a recursive approach.

## IV- FACTOR GAUSSIAN COPULA.

Andersen et al (2003) and Laurent et al (2003) amongst others have presented the one factor Gaussian copula. In this model, individual default times, which are assumed to be random and follow a normal distribution, are linked to a single common factor which is accordingly assumed to be normally distributed. From this linear dependence (i.e. the individual default times in the portfolio are linearly associated to a common factor), a correlation structure emerges between pair wise normal random default times.

A key finding here is the simplicity of computation in terms of time and expectation of portfolio loss. This is a semi-analytical expression process.

In this chapter, based on Andersen (2006) and Laurent et al (2003), we will first define the one factor copula by modeling the default time in terms of firm value conditional on the common factor. In the last section we will clearly state the distribution of the conditional default time. Our goal is to determine the number of credits that default at time t, given the value of the common factor at that time.

**IV-1 Firm's value modeling.**

So far we have dealt with the relationship between individual credit risks via their respective default time using standard Gaussian copula. In this section, given some *drawbacks* of that model, we will introduce factor framework as an improvement in terms of computation simulation and analysis.

At this point we assume that the default arrival time $\tau$ is the firm's value. For simplicity, think of this process of modeling the firm's value in terms of regression model such as CAPM (Capital Asset Pricing Model) where a certain random variable $r$ (i.e. the return on an asset), can be explained both by the level of a systematic risk $M$ (i.e. the market risk) which said to be a non-diversifiable risk and a corresponding diversifiable one say $\varepsilon$ , which depends on the firm's specific news. From this we can write a regression as follows: $r = \beta M + \varphi\varepsilon$ .

Consider here $\beta$ as the sensitivity of the asset return $r$ given the level of the systematic risk $M$ and $\varphi$ its corresponding sensitivity due to the diversifiable risk $\varepsilon$ .

Now turning to our case study, we replace the asset return by the firm's value say $Z_i$ , we keep unchanged $M$ as any common factor or the systematic risk and let $\varepsilon_i$ be the idiosyncratic risk (i.e. firm's specific news). We consider also their respective weights or sensitivities.

Note that all these variables are random and normally distributed. We can therefore write the model for the individual firm value as follows:

$$Z_i = \beta_i M + \varphi_i \varepsilon_i \qquad \textbf{(22)}$$

At this point it is important, considering a portfolio of credit risks, to highlight the fact that the correlation between pair wise idiosyncratic risks is null thereby independent. Hence $Corr(\varepsilon_i \varepsilon_j) = 0$ .

We stressed also the independency between the two sources of risk (i.e. $M$ and $\varepsilon_i$). Thus $Corr(M,\varepsilon_j)=0$ , such that the sole driver of default correlation is the common factor $M$ .

Assuming the normality assumption allows us to write that the $mean$ (i.e. the expectation) equals 0 and the $\operatorname{var}iance$ (Var) equals 1. We can then write the firm's value inb terms of expectation as $E(Z_i)=\beta_i E(M)+\varphi_i E(\varepsilon_i)=0$

And $Var(Z_i)=\beta_i^2 Var(M)+2\beta_i\varphi_i Cov(M,\varepsilon_i)+\varphi_i^2 Var(\varepsilon_i)$

Given that $Corr(M,\varepsilon_j)=0$ , $Var(M)=1$ , $Var(\varepsilon_i)=1$ . We can then write the variance as: $Var(Z_i)=\beta_i^2+\varphi_i^2$ = 1. It comes out that $\varphi=\sqrt{1-\beta_i^2}$ .

From this we can rewrite the firm value process as:

$$Z_i=\beta_i M+\sqrt{1-\beta_i^2}\,\varepsilon_i . \qquad \textbf{(23)}$$

Where $\beta_i$ implies the relationship between each firm value $Z_i$ and the common factor $M$ such that any pair wise correlation between individual firm can be written as:

$$Corr(Z_i Z_j)=\beta_i\beta_j . \qquad \textbf{(24)}$$

The next step will deal with the default probability distribution of the $Z_i$ .

**IV-2 Conditional Default probability distribution function**

At this stage, we focus on an important input in pricing a pool of credit risks. In this section before setting out the probability distribution of the default time conditional on the common factor, let us first explain what we consider as default when dealing with a firm's value.

A default occurs when a stock price of a firm drops below a certain level ***k***. From this, we can say that a credit defaults when the firm's value is below the considered barrier ***k***, that is $Z_i\le k_i$. The probability of default then is $\Pr(Z_i\le k_i)$. It is easy to make a link between this process and the Poisson process where the marginal probability distribution function of default is $\Pr(\tau_i\le t)=F_i(t)$, **equation (8).**

We can then write that $\Pr(\tau_i\le t)=F_i(t)=\Pr(Z_i\le k_i)$. At this point, as $t=\Phi^{-1}F_i(t)$, accordingly we can write that $k=\Phi^{-1}F_i(t)$ **(24) .**

Where $F_i(t) = 1 - \exp(-\lambda t)$ is the probability of default with $\lambda \left( = \frac{CDSspread}{1 - \text{Recovery}} \right)$ is the default intensity and $\Phi^{-1}$ is the inverse cumulative distribution function as stated in previous chapter.

We have now sufficient instruments to write down the conditional default probability function. Given the common factor's value, we can write the probability default distribution of the individual firm's value

$$\Pr(Z_i \le k_i / M) = \Pr\left(\beta_i M + \sqrt{1-\beta_i^2}\,\varepsilon_i \le \Phi^{-1} F_i(t) / M\right) = \Pr\left( \varepsilon_i \le \frac{\Phi^{-1} F_i(t) - \beta_i M}{\sqrt{1-\beta_i^2}} \right) \quad \textbf{(25).}$$

This can be explain by the fact that given a value of the common factor $M$ a default triggered when the firm specific information (i.e. the firm's idiosyncratic risk) hits a certain level. The cumulative distribution function of the conditional default probability that individual firm will default at a certain time t given the level of the common factor is then:

$$\Pr\left( \varepsilon_i \le \frac{\Phi^{-1} F_i(t) - \beta_i M}{\sqrt{1-\beta_i^2}} \right) = \Phi\left( \frac{\Phi^{-1} F_i(t) - \beta_i M}{\sqrt{1-\beta_i^2}} \right) \quad \textbf{(26).}$$

Where $\Phi$ is the normal cumulative distribution function.

Given the normality statement of the firm' specific information $\varepsilon_i$, we can write for computation purposes that

$$\Phi\left( \frac{\Phi^{-1} F_i(t) - \beta_i M}{\sqrt{1-\beta_i^2}} \right) = N\left( \frac{\Phi^{-1} F_i(t) - \beta_i M}{\sqrt{1-\beta_i^2}} \right) \quad \textbf{(27)}$$

Where N is the normal distribution function. We can use **NORMSDIST** for N and **NORMSINV** for $\Phi^{-1}$ function in Microsoft Excel to compute the conditional default probability, i.e. the **equation (27)**.

Note that firm's values are independent, therefore their cumulative default distribution functions are independent but there exist a pair wise correlation induce by their individual link to the common factor $M$ .

The next section will point out the marginal unconditional default probability distribution given the risk sources. This finding will allow us to defining the default correlation later in the thesis.

**IV-3 Portfolio Loss Distribution.**

The final step in implementing the one factor Gaussian copula model requires the expected portfolio loss distribution, i.e. the total loss distribution given a default event. To this end, we first set out the number of defaults since the portfolio loss distribution is a function of the loss-given-default and the number of defaults. At this point remember that the portfolio loss-given-default is: $L(k) = \sum_{i=1}^{N} l_i 1_{\tau_i \leq k_i}$ .

Where $l_i$ (i.e. 1-R) is the individual loss-given-default and $1_{\tau \leq T}$ the default arrival indicator function. The second step involves the use of the Cox process where the expectation has the ability to transform the conditional default probability into the unconditional default probability.

As we mentioned above, all the defaults in the reference portfolio are independent but for simplicity purpose let us assume that they have a unique default probability, uniform recovery rate and the all set of pair wise correlation is flat. That is we face a so called Homogeneous portfolio.

The homogenous framework implies that the average value of the conditional default probability is equal to individual default probability. We can then write the conditional default probability by dropping the marginality subscript Assume $\beta = \sqrt{\rho}$ , the conditional default probability can be written as follows:

$$\Pr(Z \leq k / M) = \Pr\left(\varepsilon \leq \frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) = \Phi\left(\frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) = N\left(\frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right). \quad \textbf{(28)}$$

Once we have defined the conditional probability of default, we can deduce the number of conditional default given different values of the common factor. Note, in such a case, where trials are identical and there is independency between them a good process to determine the appropriate number of success (i.e. the number of conditional default) is the binomial distribution**.**

Hence the probability of having ***x*** conditional a default over the process (i.e. binomial distribution) is given by:

Pr *(# of defaults=x/M)* = $B_x^n . \Pr(Z \leq k / M)^x . (1 - \Pr(Z \leq k / M))^{n-x}$ **(29)**

Where ***N*** is the number of firms in the reference portfolio. At this, it become easy to write the unconditional default probability distribution thanks to the Cox process, that is the expectation of the conditional default probability. As we will see the process involves adding up or *integrating out* theses conditional probabilities given different values of the common factor such that *M=m.*

Considering the normality of the common factor M, we can write that :

Pr *(# of defaults = x) =* $E$ ( Pr *(# of defaults = x/M=m)*

$$= E\left( B_x^N . \Pr(Z \le k/M)^x . (1-\Pr(Z \le k/M))^{N-x} \right)$$

Where $\Pr(Z \le k/M) = N\left(\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right)$, Hence,

Pr *(# of defaults =x)*

$$= \int_{-\infty}^{+\infty} \binom{n}{x} \left( N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^x \left(1-N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{n-x} \phi(m)dm.$$

Here, $\phi(m) = \frac{e^{m^2/2}}{\sqrt{2\pi}}$ is the density function of the common factor.

We can then write the unconditional default probability distribution above as:

Pr*(# of defaults =x)*

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \binom{n}{x} \left( N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^x \left(1-N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{n-x} e^{m^2/2}dm.$$

As we can see, this equation is a semi-analytical formula with no need of Monte Carlo simulation but requires a numerical process for integration purpose. This is where the Gauss-Hermte numerical integration process comes in. We ill develop the process later on in this thesis.

Once again we have sufficient inputs to compute the entire portfolio loss distribution.

Given the assumption of the homogeneous portfolio and the conditional independence of the firm's value, the average or the expected (i.e. the mean) portfolio Loss is function of the expected probability of the number of defaults.

We can therefore write that the portfolio loss-given-default distribution as a function on the number of default:

Pr *(Loss =L (k))* = Pr *(# of defaults = x*), see Kakodkar et al (2006).

Where Pr *(# of defaults = x*) is the number of default and equals to:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\binom{n}{x}\left(N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{x}\left(1-N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{n-x} e^{m^2/2}dm. \quad \textbf{(30)}$$

As we will see in a practical example throughout the next chapter, computation of the portfolio loss distribution can be implemented by combining Microsoft Excel functions and a numerical integration process.

## V- PRICING SYNTHETIC COLLATERLIZED DEBT OBLIGATION (Synthetic CDO) USING ONE FACTOR GAUSSIAN COPULA.

In this chapter, we will demonstrate how the one factor Gaussian copula can be used to price synthetic CDOs. We consider a homogeneous portfolio of credits (i.e. the default probability is constant among firms). Furthermore the independency of marginal firm values is also satisfied.

A practical example will involve the use of Microsoft Excel and the Gauss-Hermite process for numerical integration purposes.

The chapter is structured as follows: The first section deals with the definition and the mechanic of a synthetic CDO. This is where we discuss the instrument's structure and rules underlying the distribution of eventual proceeds. This section therefore state, what synthetic CDOs are, how they work and how investors use them. The second section presents the practical way to pricing a synthetic CDO under the one factor Gaussian copula. In order to fully capture the intuition behind the model, we first deal with a homogenous case before tackling a diversified portfolio of collaterals. The latter case will be discussed only in a theoretical framework.

### *V-1. Synthetic Collateralized Debt Obligation (synthetic CDOs).*

A collateralized debt obligation (CDO) is a structured investment based on a portfolio of various underlying debt instruments such as bonds (i.e. collateralized bond obligations) or loans (i.e. collateralized loan obligations) and is known as *cash* CDO.

In contrast, a synthetic CDO is an instrument where the underlying collateral is a set of credit default swaps (CDS). Furthermore, synthetic CDOs can be unfunded and are said not to be a *true sale* relative to cash CDO which are backed by real products such as bonds. Despite this difference, Elizalde (2006) pointed out the fact that both cash and synthetic CDOs are used for relatively a same target in that the latter is a replication of the former in terms of "*economic effects*".

Gibson (2004) and Andersen (2006) amongst others mentioned the growing popularity of synthetic CDOs in the credit derivatives markets. In fact investment banks, hedge funds and others institutions use CDOs for multiple purposes such as regulatory capital help, arbitrage opportunities and particularly for transferring credit risks. Furthermore, a single tranche synthetic CDO is more attractive because there is no need to deal with real assets such as bonds that is there is no need to necessarily fund the transaction.

In general, dealing with debt instruments supposes exposure to default risks. It is then obvious to assume that playing with pure credit instruments such as portfolio of credit default swaps (CDS) or synthetic CDOs, involves exposure to higher risks. In such a case, market participants are aware of how they could protect themselves against losses given the default risks embedded in their portfolio.

One way to resolve the problem is buying protection. This is not an easy task since individual investors' risk preference cannot be satisfied in terms of a portfolio of obligations.

At this point, the originator of a synthetic CDO cuts the portfolio into several slices in order to facilitate the transaction. These different parts of the pool are called tranches and range from the riskiest, (i.e. equity tranche), up to the less risky (i.e. senior tranche), via Junior and mezzanine tranches. A tranche consists of an interval, for instance $[K_1, K_2]$, such that $K_1$ represents an attachment point (lower threshold) while $K_2$ is a detachment point (upper threshold). See **Figure 2** on **Annex 2** for illustration.

As mentioned above, dealing with a synthetic collateralized debt obligation (synthetic CDO) involves buying protection against portfolio loss. This can be achieved by selling the underlying portfolio by tranches.

For example assume we have a portfolio of 100 credit default swaps. The portfolio has an initial value of £100 millions (i.e. the principal). One can divide the portfolio into four tranches that are sold to some investors.

The protection buyer, i.e. the dealer or the originator of the STDO, agrees to pay a periodic (i.e. quarterly or semi annually) fee called premium to each tranche holder until the maturity of the contract or until a default occurs. Note that the premium represents the price of the protection. On the other hand the tranche holder, i.e. the investor, agrees to pay a contingent amount should a default arise. Thus the investor agrees to take an exposure to a specific risk for various reasons with a belief that default could not occur with certain intensity. Note that the protection buyer, in such a transaction does not only transfer its credit risk but also benefit from capital regulation since the transaction is off balance sheet.

An important feature of this transaction is that the protection seller is typically betting on the future correlation of the underlying portfolio of collaterals. At the point the investors take generally the general state of the economy onto account. The following example shows how both the asset and liability sides of the portfolio are divided between investors. Given a portfolio of default swaps, such as the structure in **Figure 2**, we can summarize the corresponding proceeds as below:

| Tranche names | Tranche notional | liability rank | Cash flow waterfall | coupon if no default |
|---|---|---|---|---|
| Equity | 0-4% | 1st default | 4th payment | 40% |
| Junior | 4-9% | 2nd default | 3rd payment | 20% |
| Mezzanine | 9-12% | 3rd default | 2nd payment | 10% |
| Senior | 12-100% | 4th default | 1st payment | 5% |

Note that the tranche notional in the box above represents a fraction of the total portfolio value that the holder of the tranche (i.e. the protection seller) is responsible for, should a default occur within the life of the contract. We can see that in case of default, the investors support the portfolio loss following the seniority of their holding. Hence the first default is supported by the more risky tranche (i.e. the equity tranche) while the responsibility of the senior tranche holder is not required until the portfolio loss hits the attachment point, 12%. The cash flows distribution is in the opposite way comparing the tranches responsibility given default. A non surprising feature remains the fact that the high return (i.e. the coupon of 40%) is allowed to the more risky investment.

The coupon on the other hand represents the return associated to the investment. That is the price of the protection. This coupon is paid quarterly or semi annually until default occurs or until maturity of the transaction.

As we can see, the key in this process remains the portfolio loss distribution in that the payment of the coupon is strongly related to the default event.

Note that a synthetic CDO tranche absorbs the accumulated loss that lies in the tranche. For example, assume the portfolio faces an accumulated loss due to default of 10%. As the portfolio total loss is higher than the equity and junior tranches values in this case, the holders of these tranches then loose their total investment since the portfolio accumulated loss (10%) exceeds the 9% total loss that they are responsible for such that these tranches are completely wiped out. Consequently, no further premium should not be paid. In contrast, the senior tranche is not affected by the portfolio loss because the total portfolio loss has not hit any proportion of the tranche. That is the total portfolio loss is less than the attachment point $K_1$ of the senior tranche.

Regarding the mezzanine tranche, things are different in that the investor faces a loss of 1% of the portfolio principal. The holder of the tranche therefore looses a certain amount of its initial investment due to default. Accordingly, taking into account that default, the return (i.e. the coupon) is reduced as the notional affected to that tranche is amortized down. This idea will become clear when dealing with a practical example later on.

One of the attractiveness of a synthetic CDO is that it can be “unfunded”, i.e. the investor in this case has not to pay a notional amount at the beginning of the transaction. At this point Gibson (2004) stressed the fact the counterpart bears a credit risk that needs to be managed. One question that could arise is how do we determine the coupon value? The next section will come out with the answer.

**V-2 Pricing synthetic CDO under the One Factor Gaussian Copula: a homogeneous Portfolio case**.

In this section we use the factor Gaussian copula to price synthetic CDO tranches. When it comes to pricing a portfolio of credit risk products, key input as we said above is the portfolio loss distribution. Remember from the previous chapter that, an unconditional portfolio loss has been derived using the expectation of the Binomial distribution process that we explain as a default or not default process relative to the binary behavior of the default events. Hence the expected portfolio loss distribution is:

$$\Pr(\text{Loss=L}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\binom{n}{x}\left(N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{x}\left(1-N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{n-x} e^{m^2/2}dm. \quad \textbf{(30)}$$

Having the entire portfolio loss distribution function, we focus on a synthetic CDO tranche pricing. Gibson (2004) mentioned that to be effective any traditional collateralized debt obligation (CDO) deal requires the protection buyer to sell all the tranches in the portfolio. The credit derivatives market has seen growth of new products such as single trance synthetic CDO (STCDO). A STCDO presents some flexibility in terms of transaction in that the dealer can run the trade by selling for example only junior and mezzanine tranches and keep equity and senior tranches until he/she find an appropriate investors.

Note that the structure presented on **Figure 2** is a set of four STCDOs. At this point, let us recall the assumption that could make the tranche pricing easy to compute, that is a homogeneous portfolio of collaterals. The homogeneity implies that all the underlying credit risks have the same default probability, a uniform correlation and an identical recovery rate. An interesting finding in this case is that we approach a close form solution. Key feature of the homogenous assumption is that marginal probability of default risk can be seen as the average portfolio default probability such that the tranche loss can be calibrated to the average portfolio loss.

Hence, let us assume the tranche loss is:

$$L_{[K_1,K_2]}(t) = \max\left[\min(L(t),K_2)-K_1,0\right] \quad \textbf{(31).}$$

Where $L(t)$ is the portfolio loss.

Note that the equation above represents a percentage of the portfolio loss absorbed by the tranche. From, we can write down the value of the tranche loss as:

$$L_{[K_1,K_2]}(t) = \frac{\max[\min(L(t), K_2) - K_1, 0]}{K_2 - K_1} \quad \textbf{(32)}$$

As we can see, the portfolio loss absorbed by the tranche is a percentage of the tranche's notional. Note that $K_1$ and $K_2$ stand for the attachment and detachment points respectively.

Now given the information above, i.e. knowing the entire portfolio loss distribution as in **equation (30),** we can easily evaluate the price of a STCDO.

Recall that a STCDO swap is similar to a default swap in that, their pricing process involves a premium or fixed leg and a default or floating leg.

Remember that the tranche holder, i.e. the protection seller, agrees to pay a certain amount of capital should a default occurs. Hence the loss-given-default that the tranche holder faces and summarized in the equation **(32)** represents the floating leg at a specific time *t*.

From an investor's point of view, the payment that he could face should an default event occurs depends on the difference between the tranche loss at dates $t_{i-1}$ and $t_i$ that is the tranche loss can be written as $dL_{[K_1,K_2]}(t) = L_{[K_1,K_2]}(t_i) - L_{[K_1,K_2]}(t_{i-1})$.

Bear in mind that we are trying to set an expected amount of capital in case of default which in terms of expectation represents the tranche loss given default (also called the default leg or floating leg). This can be expressed as an expected present value of the tranche loss within the life of the trade $[0,T]$ :

$$Default\ leg(0) = E\left(dL_{[K_1,K_2]}(t)\right) = E\left(\int_0^T \exp\left(-\int_0^t r(u)du\right) dL_{[K_1,K_2]}(t)\right)$$

Here we have introduced the risk free discount factor $P(0,t) = \exp\left(-\int_0^t r(u)du\right)$.

$$Default\ leg(0) = \int_0^T P(0,t) E\left(dL_{[K_1K_2]}(t)\right)$$

Now consider that the default arises halfway between dates $t_{i-1}$ and $t_i$ as expressed in Andersen (2006). We can rewrite the above formula as:

$$Default\ leg(0) = \sum_1^N P\left(0, \frac{t_i + t_{i-1}}{2}\right) E\left(dL_{[K_1K_2]}(t)\right)$$

$$Default\ leg(0) = \sum_1^N P\left(0, \frac{t_i + t_{i-1}}{2}\right) E\left(L_{[K_1K_2]}(t_i) - L_{[K_1K_2]}(t_{i-1})\right)$$

$$Default\ leg(0) = \sum_{1}^{N} P\left(0, \frac{t_i + t_{i-1}}{2}\right)\left[E\left(L_{[K_1K_2]}(t_i)\right) - E\left(L_{[K_1,K_2]}(t_{i-1})\right)\right] \quad \textbf{(33)}$$

Regarding the premium leg of the deal, a coupon is paid on the remaining part of the tranche notional which is amortized down due to default. Thus given a default, the tranche seller, i.e. the protection buyer is no longer responsible of a full coupon payment, relative to the default free payment. At this point we can write the remaining tranche notional as:

$$(K_2 - K_1) - L_{[K_1,K_2]}(t_i) \quad \textbf{(34)}$$

Furthermore the payment of the coupon in case of default is function of the remaining number of days should default takes place between payment dates $t_{i-1}$ and $t_i$ .

As in the default leg valuation, a mathematical formulation of the premium leg can be written as :

$$\delta_i c\left[(K_2 - K_1) - L_{[K_1,K_2]}(t)\right] \quad \textbf{(35)}$$

Where $\delta_i$ the day count factor, c is represents the coupon value, $L_{[K_1,K_2]}(t_i)$ is the tranche loss at time ***t*** and $(K_2 - K_1)$ is the tranche notional.

Assume that the default occurs in midway between the payment dates. A good approximation of the remaining part following Andersen (2006) is:

$$\frac{\left[(K_2 - K_1) - L_{[K_1K_2]}(t_i)\right] + \left[(K_2 - K_1) - L_{[K_1K_2]}(t_{i-1})\right]}{2}$$

Hence the expected present value of the Premium leg can then be written as:

$$E\left(c\sum_{1}^{N} P(0,t_i)\delta_i\left(\frac{\left[(K_2 - K_1) - L_{[K_1,K_2]}(t_i)\right] + \left[(K_2 - K_1) - L_{[K_1,K_2]}(t_{i-1})\right]}{2}\right)\right)$$

$$= c\sum_{1}^{N} P(0,t_i)\delta_i\left(\frac{E\left(\left[(K_2 - K_1) - L_{[K_1,K_2]}(t_i)\right] + E\left(\left[(K_2 - K_1) - L_{[K_1,K_2]}(t_{i-1})\right]\right)\right)}{2}\right)$$

$$= c\sum_{1}^{N} P(0,t_i)\delta_i\left(K_2 - K_1 - \frac{E\left(L_{[K_1,K_2]}(t_i)\right) + E\left(L_{[K_1,K_2]}(t_{i-1})\right)}{2}\right) \quad \textbf{(36)}$$

Recall that the expectation as in the Cox process involves *integrating out* over the portfolio loss distribution.

At this point let us assume that managing the risk embedded in collaterals requires a mark-to-market process such that the premium leg and the default leg cancel each other out at any time ***t***, i.e. premium leg - default leg = 0**.**

From this the fair price of the tranche is:

$$c = \frac{\sum_{1}^{N} P\left(0, \frac{t_i + t_{i-1}}{2}\right)\left[E\left(L_{[K_1,K_2]}(t_i)\right) - E\left(L_{[K_1,K_2]}(t_{i-1})\right)\right]}{\sum_{1}^{N} P(0,t_i)\delta_i\left(K_2 - K_1 - \frac{E\left(L_{[K_1,K_2]}(t_i)\right) + E\left(L_{[K_1,K_2]}(t_{i-1})\right)}{2}\right)} \qquad \textbf{(37)}$$

Having the tranche fair price formula, we are able to price a synthetic CDO tranche applying the one factor Gaussian copula in a homogeneous portfolio framework. This will be easy since we will not use the time consuming Monte Carlo simulation.

**V-2-1 Data**

We use CMA datavision from Bloomberg. CMA datavision provides a set of daily mark-to-market data for CDS, Indices and Tranches, sourced by 30 buy side firms including Investments banks, Hedge funds and Asset managers. CMA data vision provides then the Dow Jones CDX NA IG index and the Dow Jones iTraxx EUR index, two indices which recorded the credit default swap spreads. In our thesis, we used the market quotes from Bloomberg where we focus on the Dow Jones iTraxx EUR 5 years index composed of 125 Investment grades Europeans firms and count for 24 contributors. iTraxx indices roll semi annually on March and September. The period considered is from March 20th 2007 to June 20th 2012 (i.e. the maturity date). The payments are made quarterly and the first payment has been done on June 20th 2007. The index presents 6 tranches in this order: 0-3%, 3 – 6%, 6 – 9%, 9 – 12%, 12 – 22% and 22 – 100%. Note that the index quotes allow the equity tranche to have a specific quote in that it presents an up-front fee of (33.45%), which means that the seller of protection receives (33.45%) of the tranche notional at the beginning of the trade and a quarterly premium of 500 basis points each year until default. we consider the average spreads for computation purpose. See the Annex 1 for further description.

### V-2-2 Implementation using Microsoft Excel.

For simplicity of computation purposes, we choose a hypothetic homogenous portfolio of synthetic CDO tranches. That is:

-The default probability is assumed to be the same for all firms in the portfolio.

-The correlation is assumed to be flat at 14%%

-The recovery rate is constant: 40%

-The risk-free interest rate = 5%,

-The number of firms = 125

-A 5 years contract with semi-annually payments.

The average spread of 54.27 basis points has been picked from Bloomberg. **Annex 2** presents some Bloomberg screens where one can see the structure of a contract and the Bid/Ask value of each STCDO price. **Annex 3** highlights the implementation in Microsoft Excel.

### V-2-3 The numerical Results.

The table on **Annex 4** presents our model results in terms of tranche premium. All these results are consistent with the theory of the CDO tranches pricing relative to the default correlation risk in that the premium for the most risky tranche, i.e. the equity tranche is higher than the senior tranche price. Furthermore, the higher the correlation, as discussed Gibson (2004), the higher the price of the equity tranche. In contrast the price of the senior tranche decreases when the default correlation rises. This difference in the prices' evolution as stressed in Gibson (2004) can be justified by the fact that the higher the correlation, the higher the possibility of more losses to the point that the equity, junior and mezzanine tranches can be wiped out. From this it is obvious to see that the higher the default correlation, the less the senior tranche prices. On the other hand, our results are consistent with the theory backed by Gibson (2004), which states that high correlation is a source of less default events such that the equity tranche will absorb the totality of the loss. Hence the higher the correlation, the more the equity tranche price rises.

Despite this interesting result, we have to recognize that our model quotes do not much the market prices. This can be justified by the fact that we assumed a lot of assumptions in order to ease the calculation of the Breakeven spread. It is therefore recommended to relax some of the key assumptions like the uniqueness of the default probability for the whole set of the portfolio by introducing a non homogenous framework which reflects the reality. To this end we introduce the multifactor approach as suggested by Hull and White (2004) in the next section.

## V-3 Pricing synthetic CDO under Multi Factor Gaussian Copula – a non-homogeneous Portfolio case.

We consider now the reality in the collateralized debt obligation market. That is, we relax the assumptions of the homogenous case. We therefore take into account the fact that the common traded CDOs in the industry allow difference in marginal default probabilities. Deriving the portfolio loss distribution in such a framework remains a "*stylized fact*". For instance, Laurent and Gregory (2003) used the Fast Fourier Transform (FFT) to calculate the conditional portfolio loss distribution while Andersen et al (2003) investigate a recursion approach further developed by Hull and White (2004). In this section, we aim to describe the so called "*probability bucketing*" approach introduced by Hull and White (2004) to calculate the STCDO tranches prices when marginal default probabilities are different. To this end, we rewrite in the first subsection, the asset value model and the portfolio loss distribution function. In the second subsection, we built up the *probability bucketing* followed by some comments according the approach findings in terms of synthetic CDO tranche pricing.

### V-3-1 Portfolio loss distribution under a multi factor framework.

Hull and White (2004) have considered more than one common factor in deriving the portfolio loss distribution. Thus their *multifactor copula* model also called the *Double-t distribution* is introduced. Before investigating this extension, let us recall the one factor copula framework as a building block. The firm's value in the one factor copula has been described as:

$Z_i = \beta_i M + \sqrt{1-\beta_i^2}\varepsilon_i$ or $Z_i = \sqrt{\rho_i} M + \sqrt{1-\rho_i}\varepsilon_i$ with $\beta_i = \sqrt{\rho_i}$

Where *M* and $Z_i$ are independent and $Corr(\varepsilon_i \varepsilon_j) = 0$ that is the $\varepsilon_i$ are also independent. Note also the independency between the $Z_i$ and $\varepsilon_i$ terms. All these variables are normally distributed as suggested the Gaussian copula model, (i.e. mean = 0 and Variance = 1). The time to default $t = \Phi^{-1}F_i(t)$, where $F_i(t) = 1 - \exp(-\lambda t)$ represents the probability of default and $\lambda \left( = \frac{CDSspread}{1 - \mathrm{Re}\cos very} \right)$ is the default intensity also called hazard rate $h$. At this point we noted that the cumulative distribution of the default probability is:

$$\Pr(Z_i \le k_i / M) = P\left( \varepsilon_i \le \frac{\Phi^{-1}F_i(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right) = \Phi\left( \frac{\Phi^{-1}F_i(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right)$$

$$= N\left( \frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right).$$

From this point, Hull and White (2004) introduced the *double-t distribution* or the multi factor model such that the one factor Gaussian copula became:

$$Z_i = \sqrt{\rho_{i1}}M_1 + \sqrt{\rho_{i2}}M_2 + ...... + \sqrt{\rho_{in}}M_n + \sqrt{1 - \rho_{i1} - \rho_{i2} - ...... - \rho_{in}}\,\varepsilon_i \qquad \textbf{(38)}$$

Accordingly the probability of default under one factor Gaussian copula as presented in **equation (28),** becomes:

$$\Pr(Z_i \le k_i / M_1, M_2, ..., M_n) = \Phi\left( \frac{\Phi^{-1}F_i(t) - \sqrt{\rho_{i1}}M_1 - \sqrt{\rho_{i2}}M_2 - ... - \sqrt{\rho_{in}}M_n}{\sqrt{1 - \rho_{i1} - \rho_{i2} - .......... - \rho_{in}}} \right). \qquad \textbf{(39)}$$

Hence the probabilities of zero default, i.e. the conditional probability for firm *i* to survive until trade maturity T can be written as:

$$\Pr(Z_i > k_i / M_1, M_2, ..., M_n) =$$

$$S_i(T_i / M_{1,} M_2, ......, M_n) = 1 - \Phi\left( \frac{\Phi^{-1}F_i(t) - \sqrt{\rho_{i1}}M_1 - \sqrt{\rho_{i2}}M_2 - ... - \sqrt{\rho_{in}}M_n}{\sqrt{1 - \rho_{i1} - \rho_{i2} - .......... - \rho_{in}}} \right)$$

or

$$\Pr(Z_i > k_i / M_1, M_2, ..., M_n) =$$

$$S_i(T_i / M_{1,} M_2, ......, M_n) = 1 - N\left( \frac{\Phi^{-1}F_i(t) - \sqrt{\rho_{i1}}M_1 - \sqrt{\rho_{i2}}M_2 - ... - \sqrt{\rho_{in}}M_n}{\sqrt{1 - \rho_{i1} - \rho_{i2} - .......... - \rho_{in}}} \right) \qquad \textbf{(40)}.$$

Note that we replace $\Phi$ by *N* (i.e. the normal distribution indicator) for computation purposes, while $k_i$ remains the default barrier.

Note that the equation of the conditional survival probability above can be written as:

$$\Pr\left(Z_i > k_i / M_1, M_2, ..., M_n\right) = \prod_1^n S_i\left(T_i / M\right),$$

From this, the unconditional portfolio loss distribution can be written as an integration process:

Pr(Portfolio Loss) = $\int_{-\infty}^{+\infty} \prod_{i=1}^{n} S_i\left(T_i / M\right)\varphi(m)dm$ **(41).**

Where $\varphi(m) = \frac{1}{\sqrt{2\pi}} e^{m^2/2}$ is the density function of the common factor ***M.***

This is where Andersen et al (2003) used a recursive approach to implement the portfolio loss distribution further developed by Hull and White in a probability bucketing framework.

### V.3-.2 Hull and White "probability bucketing"

In this subsection the so called *probability bucketing* approach is described. Note that the assumptions in the homogenous case are partly relaxed. Thus we consider for this approach that:

-the firms in the portfolio have different default probabilities and are independent within the time period on the trade.

-the loss-given-default (i.e. $LGD_i = 1 - R$) for individual firms are identical since the t recovery rate is constant.

Hull and White (2004) themselves recognized that the probability bucketing approach has been previously implemented in a recursive framework by Andersen et al (2003) where the key feature is to add firms into the portfolio one by one. Hence based on this latter technique, Hull and White (2004) divided first the total potential loss of the portfolio into a large number of small intervals called buckets, such that we have $\{0, b_0\}, \{b_0, b_1\}, .................\{b_{k-1}, \infty\}$ which represent respectively the 0*th*, 1*st* up to the K*th* buckets. From this we can say that each tranche of the CDO involves a certain number of small intervals or buckets.

Note that the probability bucketing approach considers two kind of probability such that the portfolio loss distribution can be implemented based on:

- $P_T(k)$, *the probability that the loss by time T lies in the bucket k*

- $P(k,T)$*, the probability that the loss by time T lies in the bucket k or in the higher one*.

The key point is therefore to calculate the probability that the loss by time **T** lies in the bucket *k,* conditional on the common factor's realization.

At this point we assume that $\pi_T(k)$ represents the probability of exactly k defaults in the portfolio. Hence the portfolio survival probability (i.e. the case of zero default by time T) can be written as:

$$\pi_T(0) = \prod_{i=1}^{n} S_i(T_i / M_1, M_2, ....., M_n) \qquad \textbf{(42)}$$

From this, Hull and White assumed that the probability bucketing approach is essentially adding up the firms into the portfolio one by one such that the entire portfolio loss distribution is deduced.

Note from equation **(41)** that the probability of default relative to the second bucket $b_1 = 1$ can be written as a function of the previous bucket. That is :

$$\pi_T(1/M_1, M_2, ......, M_n) = \pi_T(0/M_1, M_2, ......., M_n) \sum_{i=1}^{N} \frac{1 - S_i(T/M_1, M_2, .....M_n)}{S_i(T_i/M_1, M_2, .....M_n)}$$

Accordingly the probability that the loss by time T lies in the bucket $b_1 = 1$ can be written as:

$$P_T(1/M_1, M_2, ......, M_n) = P_T(0/M_1, M_2, ......., M_n) \sum_{i=1}^{N} \frac{1 - S_i(T_i/M_1, M_2, .....M_n)}{S_i(T_i/M_1, M_2, .....M_n)} \qquad \textbf{(43)}$$

The unconditional probability is a function of a numerical integration other each value of M. Thus the total portfolio loss can be written as:

$$P(n,T) = \int \prod_{k=n}^{K} P_T(k) \varphi(m) dm \qquad \textbf{(44)}$$

Where $\varphi(m) = \frac{1}{\sqrt{2\pi}} e^{m^2/2}$ is the density function of the common factor ***M.***

Note that Hull and White estimated that the probability that the first loss happens in the mid point of a bucket, (i.e. $0.5(b_{k-1} + b_k)$ ) between the time interval of time $[T_1, T_2]$ can be written as: $P_T(k) = 0.5[P(k,T_2) + P(k+1.T_2)] - 0.5[P(k,T_1) + P(k+1,T_1)]$ **(45)**

From this point the authors used the individual loss distribution to compute the portfolio loss distribution by numerical integration, i.e. the Gauss-Hermite can be used. Once the portfolio loss is found we use the **equation (37)** to calculate the STCDO price, (i.e. the premium).

$$c = \frac{\sum_{1}^{N} P\left(0, \frac{t_i + t_{i-1}}{2}\right)\left[E\left(L_{[K_1,K_2]}(t_i)\right) - E\left(L_{[K_1,K_2]}(t_{i-1})\right)\right]}{\sum_{1}^{N} P(0,t_i)\delta_i\left(K_2 - K_1 - \frac{E\left(L_{[K_1,K_2]}(t_i)\right) + E\left(L_{[K_1,K_2]}(t_{i-1})\right)}{2}\right)}$$

## VI- RISK ANALYSIS OF SYNTHETIC CDOs.

So far we have defined a process to price a single tranche synthetic CDO (STCDO). Thanks to the Gaussian copula model the marginal default probability distributions and their pair wise correlation have been bound and to produce with a join distribution that led to a price of a STCDO. From this point, we realised that pricing correlation dependent structured products is an art that is strongly related to the feeling of the market participants in terms of model implementation in that different default correlation level emerged due to different stylised methods. In this chapter, we perform an analysis of the risk underlying the market view of the participants, which they explain through the factor Gaussian copula model. To this end we will first explore the factor Gaussian copula model assumptions in comparison with the well known Black-Scholes model, another benchmark model that leads the equity and interest rate options markets. Here we will assess a definition of the implied correlation. In the second section we will analyse the risk embedded in each single tranche of a synthetic CDO relative to the level of the correlation and the strength of the economy. Next, we go on by introducing the “base correlation” and its impact on both the STCDOs pricing and the risk analysis that involves. We terminate the chapter by a risk analysis of the recent Bears Stearns’ funds collapse.

### VI.1 Assumptions

Remember that in the Black-Scholes model, the sole unobservable variable is the volatility which expresses the global view of market participants (i.e. their forecast). As such, the volatility can be seen as the price driver in equity and interest rates markets.

Thus, given the market price of an option, and fixing the others inputs, (i.e. interest rate, strike price, stock price, time to maturity) one is able to calculate the volatility. That is called *implied volatility.* From this, it is obvious that if the Black-Scholes model is correct, the implied volatility must be equal to the one that produces the market price as mentioned Elizalde (2005).. In fact, the Black-Scholes model considers that the distribution of an asset return is lognormal as highlighted in McGinty et al (2004). As a result, the mean is zero and the variance is one. This statement implies that the error term is very small. Note that given the assumption of normal distribution the resulting implied volatility, should necessarily be the same for all the market participants. Unfortunately, the implied volatility curve displays a skew that definitely proves that the *benchmark* model even widely used in the equity and interest rates market is not realistic.

Now by analogy, think of the factor Gaussian copula model in terms of the model describes above. Thus the price of the single tranche synthetic CDO (STCDO) can be compared to the market price of the option. At this point the correlation of default as the main price driver is similar to the volatility such that the implied correlation is identical to the implied volatility. Given the normality assumption of the firm's value as implied by the factor Gaussian copula model, the implied correlation, should be the same for all the tranches of the portfolio. Once again, as in the Black and Scholes model, the implied correlation that drives the current market price in the factor Gaussian copula framework is different for each tranche of the synthetic CDO. Hence the supposed flat correlation underlying the entire portfolio of STCDO does not match the reality. For instance Elizalde (2005) highlighted the fact that "*the implied correlation is higher for equity and senior tranches than for mezzanines tranches"*. We can then argue that the implied correlation is a function of the attachment and detachment points such that any "exotic" tranche can not be created to satisfied investors who would like to be exposed to some tailored tranche risk other than those provided in the CDX NA IG and the Dow jones iTraxx EUR. This is where McGinty et al (2004) introduced the "base correlation" that is said to resolve the problem of pricing non standard CDO tranche. For more definition on base correlation see the McGinty (2004) and Galiani et al (2006).

Note that both implied and base correlations present a smile (i.e. a skew). Hence, the diversity of the implied default correlation as a result implies that the market participants have different view of default co-dependency. This is where arbitrage opportunities arise, which lead to some risk management needs. Given the definition of implied correlation above, let us move onto the analysis of risk that involves.

**VI-2 Synthetic CDO Risk Analysis.**

Trading credit derivatives and particularly synthetic CDOs without completely capturing the risk embedded in, can lead to some trouble in terms of huge capital losses. This is evident when considering implied correlation outputs as an expression of investors' beliefs in terms of correlation modelling. Hence trading CDO tranches in general and particularly synthetic CDO can simply be a bet. As such, any investor can be wrong.

Accordingly, David Li (i.e. the pioneer of the Gaussian copula model) argued as reported the Wall Street Journal (2005) *“..... The most dangerous part of the model is when people believe everything coming out of it..... Investor who put too much trust in it or do not understand all its subtleties may think they have eliminated their risks when they have not...”*

We can explain this as the fact that the pricing model is nothing but what one's belief such that the reality is not necessarily matched. Hence apart from the traditional risk inherent to any debt contract, synthetic CDO tranches bear couple of risks relative for example to the model in use, the risk transfer process the default correlation between underlying asset values and particularly the impact of the level of the economy.

In this section, thanks to Gibson (2004), we explain the risks that the synthetic CDO tranches involve. We then examine by analogy the risks implied by the “*subprime mortgages backed securities*” based investments that Bears Stearns is suffering from. At this point we focus on single tranches of a synthetic CDO as in the structure described on the Dow Jones iTraxx EUR index in chapter 5. We consider first the ability for each tranche to effectively transfers risks from the product's originator to a potential investor. Gibson (2004) has demonstrated that the equity tranche, as the riskiest one, supports most of the portfolio risk, to the point that, keeping this tranches for any reason even for "*moral hazard*" (i.e. investors usually prefer the creator of CDO, to have his/her money exposed to the first range of risk) when selling synthetic CDO tranches, is actually equivalent to transferring a very small amount of risk. The second point we would like to stress is the risk relative to the level of the default correlation. At this point remember that the tranches market prices reflect the market participant's view of the underlying assets default co-dependency. As such a high default correlation could be a guarantee of a strong stability which can imply few defaults occurrences within the portfolio**.** On the other hand, we agree by intuition that a high default correlation bears a dramatic impact to the point that the magnitude of the expected loss that the portfolio suffers could reach the senior tranche having previously wiped out the equity and the intermediate tranches**.**

From this, an interesting figure has been drawn by Gibson (2004). The author demonstrated how much the equity and mezzanine tranches bear a leverage risk exposure should a default occur (i.e. the losses could reach five times those of the underlying assets)**.** Note that the equity and the senior tranches are very sensitive to the default correlation, while the mezzanine tranches in contrast are strongly related to the business cycle, see Gibson (2004).

Now, let us going on to assess the risk exposure given the strength of the economy. To this end remember that the default correlation of the underlying asset values in the portfolio has been defined as the pair wise relationship between firm's values relative to the common factor which is argued to be the general state of the economy (also called the business cycle).

At this point let us introduce and examine the Bears Stearns hedge funds collapse. Bears Stearns is an investment banks in trouble after making a *wrong bet* on some highly risky mortgages-Backed-Securities (MBS). In fact, the investment bank has formed two *high-risk, speculative investment vehicles* (see Reuters, 2007) to deal with the riskiest housing market in the United States. The structure of the deal can be depicted as follows: in the United States, some lenders provided to risky borrowers the total of the capital that they need to buy their houses. In turn these lenders bought protection against the default from these borrowers by packaging their mortgages as bonds and selling them to some investors. The sellers of protection (i.e. the buyers of bonds backed by mortgages) now hold a portfolio of securities called Mortgages-Backed-Securities (MBS).

The holders of these MBS obviously transfer the risk embedded in their underlying mortgages by repackaging the portfolio. That is they slice these low grade instruments (i.e. very risky investments) into tranches. These CDO tranches are then sold to a third party, for example to another investment bank.

By analogy to the synthetic CDO tranches, these new instruments, i.e. the MBS, present some investment-grades features. At this point one may pay attention to the fact that a very risky investment is being transformed in a low risk investment. Note that this mechanism is nothing in terms of intrinsic value but a result of the market participants' willing. At this point, we can link the Bears Stearns' funds collapse to different risk exposure. To this end, we assume that Bears Stearns is the dealer who bought bonds based mortgages and then sold the proceeds in tranche to other investments banks. From this the first source of the trouble could be the fact that Bears Stearns as tranche seller retained the equity tranches in the transaction. In fact if the case is verified, the investment bank actually faces a leverage risk relative to the equity and mezzanine tranches risk exposure, because as explained by Gibson (2004), these tranches bear the essential of the portfolio risk while only the essential of the portfolio notional has been transferred. Another source of the Bears Stearns' funds collapse is certainly linked to the fact that the investment bank has borrowed the entire capital invested in buying the "bonds" backed by the mortgages. That is the resulting hedge funds are exposed to other leverage (i.e. a part from the leverage exposure faced by the equity and mezzanine tranches) effects to the point that the default in the US second housing led to losses of at least five times the loss suffered by the sellers of the bonds backed by mortgages.

Regarding now to the third party' side, investors obviously face not only the risk due real correlation level (by opposition to their forecast), but also due to the negative impact of the business cycle. Reuters (2007) reported that "Bear Stearns has created MBS based CDOs in 2005 and 2006. At that time the United States' housing market was in its late upward move. Thus the general state of the economy was high. The resulting inflation was then at a top level. This is where the monetary authorities usually act by increase the interest rate in order to slowdown the inflation.

The downside of such a politic unfortunately is that, the wages drop dramatically. As a result, as mentioned in the report cited above, more than 5 million people who had borrowed hundred percent of the value of their house failed to pay back. Hence the underlying Mortgages-Backed-Securities then collapse. This is where the key indices such as Dow Jones, FTSE 100, and CAC40 among others became nervous due to second housing market trouble in the United States of America.

## VII-CONCLUSION.

This thesis has been dedicated to a detailed presentation of the one factor Gaussian copula model and the probability bucketing approach as an extension which is said to ease the computation of single tranche synthetic CDOs prices in terms of time. To this end, we used many researches for correlation products modelling, for instance Laurent and Gregory (2003), Andersen et al (2003), Andersen (2006) and Hull and White (2004)). We then compare the STCDOs tranches prices under the one factor Gaussian copula given a homogenous portfolio assumption with the market quotes. We found that such an implementation is appealing in that the theoretical issue about the difference in the level of the tranches prices is respected. However the model results are actually far from the market quotes.

From this, we explained the extent to which the resulting treatment of correlation for single tranche synthetic CDOs (STCDO) can be understood to capture the risks that market participants are actually dealing with. At this point we noted that the default correlation, as the main price driver, is essentially a result of markets participants' beliefs. As such we pointed out the difficulty behind the interpretation of the default correlation which display some skew generated by the implied correlation curve. Furthermore, we reported the "base correlation" framework, an attempt by McGinty et al (2004) to resolve the problem underlying the implied correlation in terms of pricing non standard tranches such as CDO squared and further developed by Hull and White (2004), in their probability bucketing approach.

Based on Gibson (2004) we also assess an analysis of the risk embedded in the synthetic CDO tranches. Hence from the credit dealer' side, we reported the fact that the reality of a transaction, in terms of credit risk transfer, when keeping the equity tranche, is of nothing but transferring the largest part of the portfolio notional. This is because the essential of the risk remains in the equity tranche. At this point we have stressed the leverage exposure of the equity and the mezzanine tranches as mentioned by Gibson (2004).

On the other side of the deal, it has been clearly stated that the investor has to be aware how much default correlation can impacts in various manner the risk in each tranche. Finally, referring to the Bears Stearns trouble case we pointed out couple of risk due to leverage investment and especially to the business cycle which definitely drives the risk embedded in the synthetic CDOs tranches as correlation based products.

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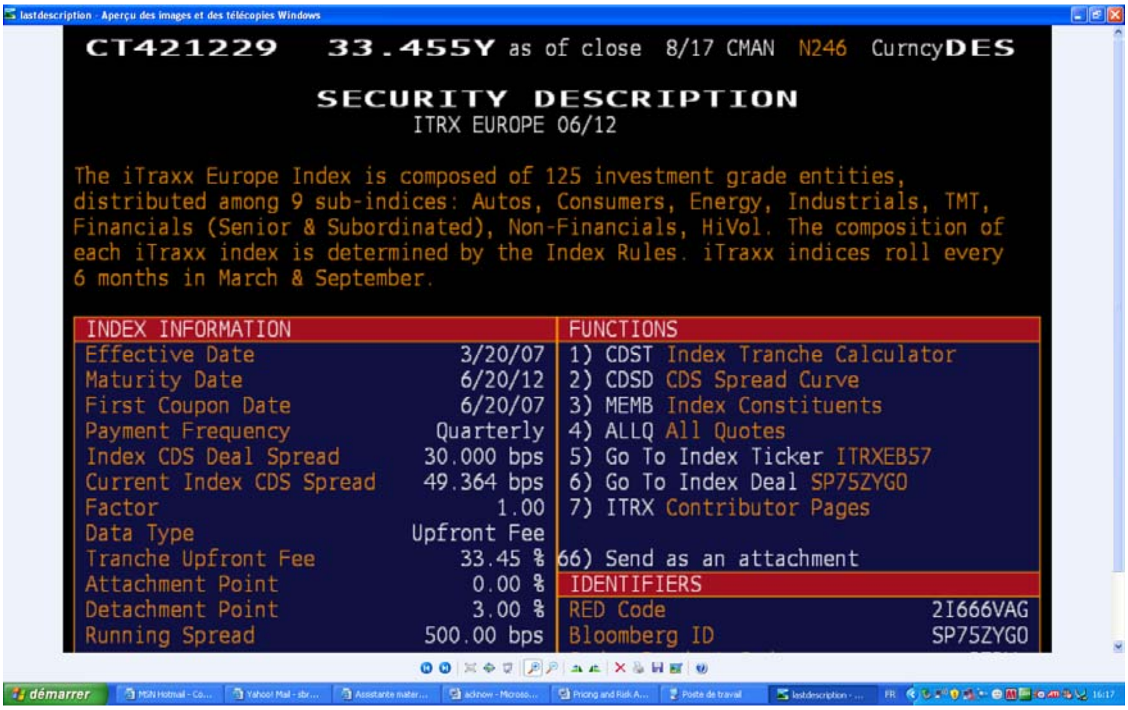
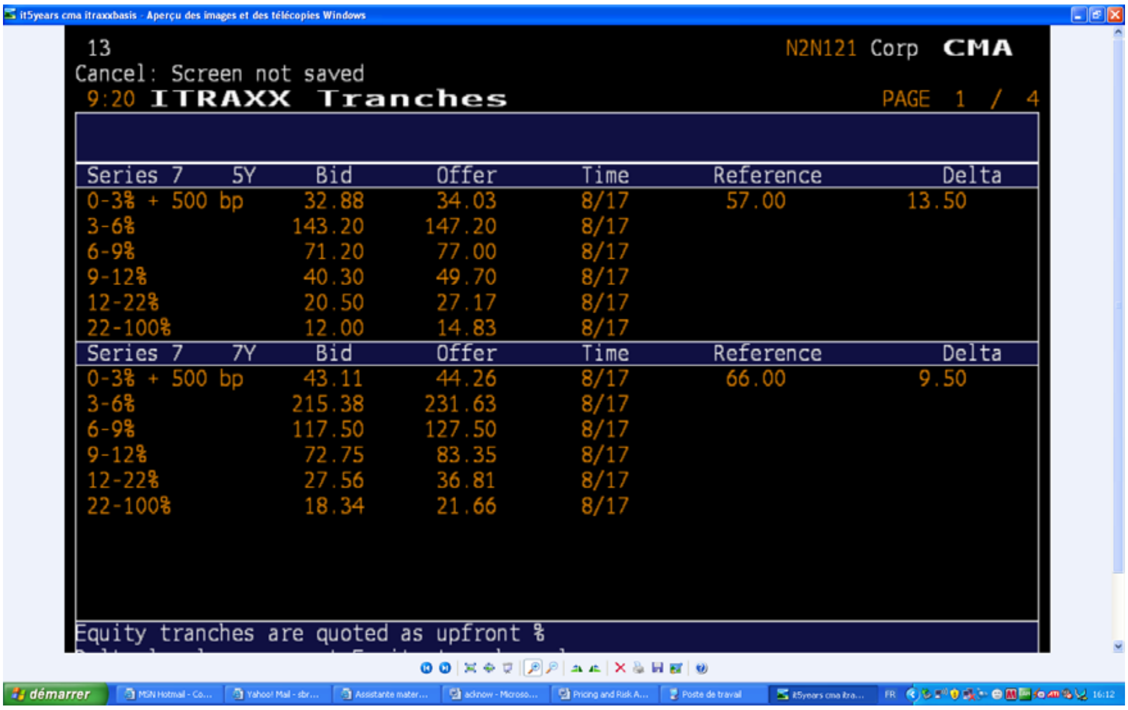
# Annex I




***Source: Bloomberg. The structure of a STCDO contract***




***Source: Bloomberg. Bid/Ask values of the STCDO swap.***

## Annex 2

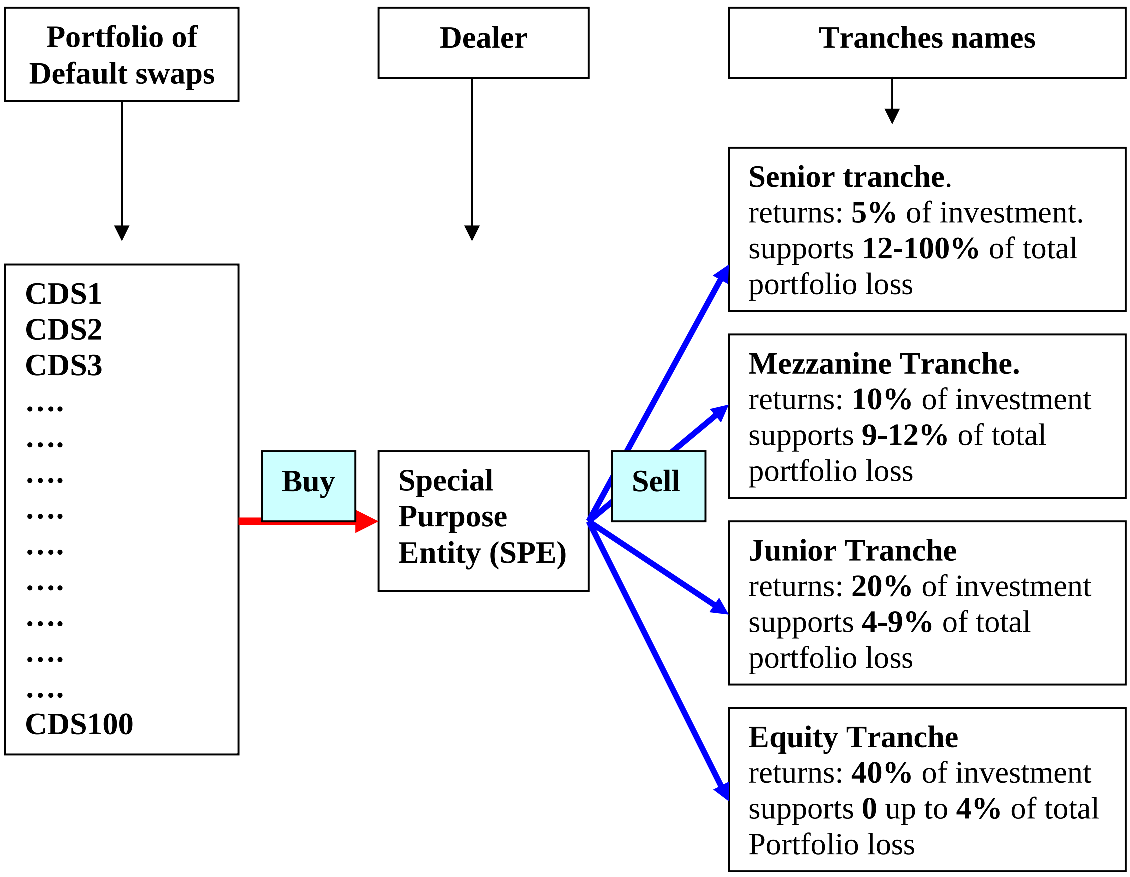



**Figure 2. The Structure of a synthetic CDO**

**Annex 3**

We implement a single tranche synthetic CDO pricing. We use the one factor Gaussian copula framework in Microsoft Excel.

The STCDO pricing process in **Section V-2** can be spanned into five steps as follows:

**Step1.**

We compute the homogenous portfolio loss distribution. Given the description of the portfolio, this step includes **2** points. The key input is the probability of default which is resumed in equation 8, that is the first point:

. $F(t) = 1 - S(t) = 1 - \exp(-\int_0^t \lambda(u)du$ Where $\lambda = \frac{spread}{1 - \mathrm{Re}\,\mathrm{cov}\, eryRate}$

In Excel we can write down: **= 1 – Exp(-(Spread/1-Recovery Rate)*Time).**

The second point is to applying both the binomial distribution and a numerical integration to find out the portfolio loss distribution. To this end we used the Gauss-Hermite numerical integration. Our goal here is to compute the formula below as the portfolio loss distribution.

Pr(Loss=L)= $\int_{-\infty}^{+\infty} \binom{n}{x} \left( N\left[ \frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right] \right)^x \left( 1 - N\left[ \frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right] \right)^{n-x} \phi(m)dm.$

We can see in this case that the first point to be computed is the binomial distribution over different values of the common Factor M. that is the conditional portfolio loss.

In Excel we first compute:

$$\left( N\left[ \frac{\Phi^{-1}F(t) - \sqrt{\rho}M}{\sqrt{1-\rho}} \right] \right)$$

**=NORMSDIST ((NORMSINV (Default Probability)-SQRT (Correlation)*Value of the common factor)/SQRT (1-Correlation))**

Assume the formula above equals: **CDP** (conditional default probability)

And then the binomial distribution as:

**= (BINOMDIST (number of defaults, PortfolioSize, CDP), FALSE).**

We will replace the whole formula by: **“BIN”** next time for simplicity.

The last point involves combining the Binomial distribution with the Gauss-Hermite numerical integration to find out the Unconditional Portfolio loss distribution. To this end we used the formula below from:

***http://www.efunda.com/math/num_integration/findgausshermite.cfm***

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} e^{-x^2}\left[e^{x^2} f(x)\right]dx \approx \sum_{k=1}^{N} w(x_k)e^{x_k^2} f(x_k)$$

Where $w(x_k)$ represents the weights of the density function $e^{X_k^2}$ common

This is the Gauss-Hermite numerical integration to estimate. It can be compared to the equation

Pr(Loss=L)= $\int_{-\infty}^{+\infty}\binom{n}{x}\left(N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^x\left(1-N\left[\frac{\Phi^{-1}F(t)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right]\right)^{n-x}\phi(m)dm.$ **(30)**

Hence we compute this formula in Excel as:

**=SUM ("BIN"*NORMDIST (Common Factor value, 0, 1)*the corresponding weight value)**

We will refer to this formula as **UPL**, i.e. unconditional portfolio default probability. it represent the number of default . This result will be referred as **UPL** i.e. the portfolio loss given default

We have to repeat all the above steps for all the time point within the 5 year period and for all firms in the portfolio.

**Step 2**

The second step is an input to computing the expected loss of the tranche in the next step. Hence in this step we compute the tranche loss using the tranche loss function as in equation **(32):**

$$L_{[K_1,K_2]}(t) = \frac{\max[\min(L(t), K_2) - K_1, 0]}{K_2 - K_1}$$

In Excel sheet we enter = **MAX (MIN (UPL, K2) - K1, 0) / Tranche notional.** This formula is entered for each time period and for each firm.

We will refer to this formula as **TL**, i.e. tranche loss

**Step 3**

This is where we combined the result in step 1 and step 2 for each time period and for each firm as well, as a result we have the Expected tranche loss as the sum of product of the unconditional portfolio loss and the tranche loss for each time period.

In Excel then, for each time period we enter **= UPL(i)*TL(i)** where i represents each firm.

**Step 4**

We repeat all the steps above for the eleven (11) time points required for the 5 year contract.

**Step 5**

The last step involves the calculation of the present value for the default leg and the premium leg before dividing the results as in equation **(38).**

$$Default\ leg(0) = \sum_{1}^{N} P\left(0, \frac{t_i + t_{i-1}}{2}\right)\left[E\left(L_{[K_1 K_2]}(t_i)\right) - E\left(L_{[K_1,K_2]}(t_{i-1})\right)\right] \qquad \textbf{(33)}$$

*Premium leg =*

$$c\sum_{1}^{N} P(0,t_i)\delta_i\left(K_2 - K_1 - \frac{E\left(L_{[K_1,K_2]}(t_i)\right) + E\left(L_{[K_1,K_2]}(t_{i-1})\right)}{2}\right) \qquad \textbf{(36)}$$

Here we just define the computation of the date and then the discount factor in Excel.

The **Date**, i.e. each time point can be defined as:

**=DATE(YEAR(the date before now), MONTH (the date before now)+6("this is because of the semi annually payment),DAY(the date before now).**

At this point the time, i.e. the number of days between two time periods is the time that we consider in computing the Discount rate as follows:

**= 1/ (1+ Risk free rate)^time.**

As we can see, calculating the premium leg require the fraction of the remaining time when default occurs before the next payment date.

In Excel we compute this as year fraction.

**=YEARFRAC ("Start Date", the current date, 3).**

Note that the 3 in the function above represents the day count convention which is 365 in our case.

After all these calculations we used the formula:

$$c = \frac{\sum_{1}^{N} P\left(0, \frac{t_i + t_{i-1}}{2}\right)\left[E\left(L_{[K_1,K_2]}(t_i)\right) - E\left(L_{[K_1,K_2]}(t_{i-1})\right)\right]}{\sum_{1}^{N} P(0,t_i)\delta_i\left(K_2 - K_1 - \frac{E\left(L_{[K_1,K_2]}(t_i)\right) + E\left(L_{[K_1,K_2]}(t_{i-1})\right)}{2}\right)}$$

That is the fair price of the single tranche synthetic CDO.

## Annex 4

| Dow Jones iTraxx Europe | | | | | | |
|---|---|---|---|---|---|---|
| **Tranche** | **0 – 3%** | **3 – 6%** | **6 - 9%** | **9 - 12%** | **12 - 22%** | **22 - 100%** |
| **Market Quotes (Mid points)** | **33.66%** | **145.2** | **74.1** | **45.05** | **23. 84** | **14.42** |
| **Correlation** | **Model Quotes** | | | | | |
| **14%** | **47.28%** | **46.06** | **44.72** | **43.44** | **11.05** | **0.40** |
| **20%** | **50.48%** | **49.05** | **49.38** | **51.48** | **10.06** | **0.22** |
| **25%** | **54.56%** | **54.06** | **55.32** | **56.23** | **8.80** | **0.15** |
| **30%** | **59.06%** | **58.93** | **59.53** | **57.97** | **7.78** | **0.13** |
| **40%** | **65.52%** | **63.77** | **60.66** | **55.93** | **6.47** | **0.10** |

**Single Tranche Synthetic CDO Pricing: a homogenous Portfolio under One Factor Gaussian Copula**