Option on the Spread between two assets, *Horacio Aliaga (Nov 2014)*

The payoff on the spread between assets R and B is defined as:

$$(R - B - K)^{+}$$

Where K is the strike and + denotes flooring to zero

We will need to integrate a vector of normally distributed variables:

$$\overline{X} = (X_1, X_2)$$

$$X_1 \sim N(0,1)$$

$$X_2 \sim N(0,1)$$

The covariance matrix is:

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Which has as determinant:

$$\det(\Sigma) = (1 - \rho^2)$$

and inverse:

$$\Sigma^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

The derivation below will be used to separate the joint distribution into unconditional and conditional distributions:

$$\overline{X} \cdot \Sigma^{-1} \cdot \overline{X}^{T} = \frac{1}{(1-\rho^2)} \left({X_1}^2 - 2\rho X_1 X_2 + {X_2}^2 \right)$$

$$= {X_2}^2 + \frac{1}{(1-\rho^2)} (X_1 - \rho X_2)^2$$

On the other hand the Refined (R) and Base (B) underlyers follow the processes:

$$R = R_{0,T} \mathrm{Exp}(-\frac{{\sigma_1}^2 T}{2} + \sigma_1 \sqrt{T} X_1)$$

$$K = K_{0,T} \mathrm{Exp}(-\frac{{\sigma_2}^2 T}{2} + \sigma_2 \sqrt{T} X_2)$$

So the undiscounted value of the spread option in terms of the joint distribution is:

$$E_{(X_1,X_2)}[(R-B-K)^+] = \iint_{-\infty}^{+\infty} \frac{dX_1 dX_2}{2\pi\, \det(\Sigma)^{1/2}} Exp\left(-\frac{1}{2}\overline{X}\cdot\Sigma^{-1}\cdot\overline{X}^T\right)[(R-B-K)^+]$$

This can be expressed in terms of conditional expectations:

$$E_{(X_1,X_2)}[(R-B-K)^+] = E_{X_2}\left[E_{X_1}[(R-B-K)^+]|X_2\right]$$

Explicitly in the form:

$$E_{X_2}\left[E_{X_1}[(R-B-K)^+]|X_2\right] =$$

$$\int_{-\infty}^{+\infty} \frac{dX_2}{2\pi^{1/2}} Exp\left(-\frac{{X_2}^2}{2}\right)\int_{-\infty}^{+\infty}\frac{dX_1}{[2\pi(1-\rho^2)]^{1/2}} Exp\left(-\frac{(X_1-\rho X_2)^2}{2(1-\rho^2)}\right)[(R-K')^+]$$

fixingX_2, we can fix a new strike:

$$K' = K_{0,T}\exp\left(-\frac{{\sigma_2}^2 T}{2}+\sigma_2\sqrt{T}X_2\right)+K$$

And doing the change of variables

$$X = \frac{X_1-\rho X_2}{\qquad}$$

We can express the previous expression as:

$$E_{X_2}\left[E_{X_1}[(R-B-K)^+]|X_2\right] =$$

$$\int_{-\infty}^{+\infty} \frac{dX_2}{2\pi^{1/2}} Exp\left(-\frac{{X_2}^2}{2}\right)\int_{-\infty}^{+\infty}\frac{dX}{[2\pi]^{1/2}} Exp\left(-\frac{X^2}{2}\right)\left[(R_{0,T}\exp(-\frac{{\sigma_1}^2 T}{2}+\sigma_1\sqrt{T}(\sqrt{(1-\rho^2)}X-\rho X_2) - K')^+\right]$$

(Notice the additional drift of the refined process). For the K' integral the lower integration boundary is:

$$X_{inf} = \frac{Ln\left(\frac{K'}{R_{0,T}}\right)-\rho\sigma_1\sqrt{T}X_2+\frac{{\sigma_1}^2 T}{2}}{\sigma_1\sqrt{(1-\rho^2)}}$$

So the related inner integral is:

$$\int(2) = -K'\cdot N\left(\frac{Ln\left(\frac{R_{0,T}}{K'}\right)+\rho\sigma_1\sqrt{T}X_2-\frac{{\sigma_1}^2 T}{2}}{\sigma_1\sqrt{(1-\rho^2)}}\right)$$

For the inner integral of the refined process, we need to complete squares:

$$-1/2\left\{X^2+\sigma_1{}^2T-2\sigma_1\sqrt{T(1-\rho^2)}X-2\rho\sigma_1\sqrt{T}X_2\right\}=$$

$$=-\frac{\left\{\left(X-\sigma_1\sqrt{T(1-\rho^2)}\right)^2+\sigma_1{}^2T-\sigma_1T(1-\rho^2)-2\rho\sigma_1\sqrt{T}X_2\right\}}{2}$$

$$=-\frac{\left(X-\sigma_1\sqrt{T(1-\rho^2)}\right)^2}{2}-\frac{\sigma_1{}^2T\rho^2}{2}+\rho\sigma_1\sqrt{T}X_2$$

So the lower integration boundary is:

$$\hat{X}_{inf}=\frac{Ln\left(\frac{K'}{R_{0,T}}\right)-\rho\sigma_1\sqrt{T}X_2-(1-\rho^2)\sigma_1{}^2T+\frac{\sigma_1{}^2T}{2}}{\sigma_1\sqrt{(1-\rho^2)}}$$

And the refined inner integral is:

$$\int(1)=R_{0,T}\mathrm{Exp}(-\frac{\sigma_1{}^2\rho^2T}{2}+\sigma_1\rho\sqrt{T}X_2)\cdot N\left(\frac{Ln\left(\frac{R_{0,T}}{K'}\right)+\rho\sigma_1\sqrt{T}X_2-\rho^2\sigma_1{}^2T+\frac{\sigma_1{}^2T}{2}}{\sigma_1\sqrt{(1-\rho^2)}}\right)$$

Then total integral over Black Scholes-like functions is:

$$\int_{-\infty}^{+\infty}\frac{dX_2}{2\pi^{1/2}}Exp\left(-\frac{X_2{}^2}{2}\right)\left\{\int(1)+\int(2)\right\}$$

Which can be put in direct terms of:

$$\int_{-\infty}^{+\infty}dYe^{-Y^2}\left\{\frac{e^{\frac{Y^2}{2}}}{\sqrt{2\pi}}\left\{\int(1)+\int(2)\right\}\right\}$$

In order to be readily for integration using the Gauss Hermite Quadrature scheme