Pricing exotic options under local volatility

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December 13, 2004

Abstract

In this paper, we apply the Jensen's inequality to derive sharp lower and upper bounds for conditional expectations of multiplicative functionals of diffusion processes. We show that the bounds are applicable to the pricing of European-style derivative securities. We also adapt the method to price (double) barrier options and lookback options under local volatility models. As by-product result, we propose a static hedge for lookback options. Numerical illustrations performed for the CEV model demonstrate the high accuracy of the method.
Keywords: Jensen's inequality; Local volatility; Lookback and barrier options; Static hedging

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1 Introduction

Consider a security that guarantees a payment $h(.)$ contingent on the state of the process X at maturity date t. From arbitrage free arguments, we can determine the price of a contingent claim as the expectation under some risk-neutral measure $\mathbb{Q}$ of the discounted payment

$$V(x,t) = E_x^{\mathbb{Q}}\left[e^{-\int_0^t r(X_s)ds} h(X_t)\right] \tag{1}$$

where $X_0 = x$ and $r(x)$ is the instantaneous discount rate. For a call option $h(x) = (x - K)_+$ where K is called the strike price. If the underlying asset is traded and the discount rate is the constant risk-free rate r, the risk-neutral dynamic of the asset is

$$\frac{dX_t}{X_t} = (r-d)dt + \tilde{\sigma}(X_t,t)dW_t, \quad X_0 = x \tag{2}$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion and d is the dividend rate. The function $\tilde{\sigma}(x,t)$ is called the local volatility. By standard application of Feynman-Kac theorem, we obtain that $V(x,t)$ is the solution of the so-called pricing partial differential equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}x^2\tilde{\sigma}^2(x,t)\frac{\partial^2 V}{\partial x^2} + (r-d)x\frac{\partial V}{\partial x} - rV \tag{3}$$

subject to the condition $V(x,0) = h(x)$. The local volatility function is assumed to be constant in the Black-Scholes setting. Nevertheless, empirical evidences suggest that the volatility implied by the price of traded options varies with the maturity and the strike price. The feature that the implied volatility is higher for low strike is referred in the literature as the volatility skew. The term volatility smile relates to implied volatility shapes with a minimum for a particular strike. Alternative specifications of the local volatility were introduced to allow for the volatility skew and smile. For further details on the relation between local and implied volatility, we refer to Dupire (1994) and to Derman and Kani (1996).

For arbitrary specification of the local volatility, no closed form expression exists for the price $V(x,t)$. However, local volatility models offer a tractable alternative to the Black-Scholes formula that allows to price exotic derivatives consistently with plain vanille option market data. Indeed, Dupire (1994) was able to calibrate analytically the local volatility surface to the market through the formula

$$\tilde{\sigma}^*(K,t) = \sqrt{2\frac{\frac{\partial V}{\partial t} + (d-r)K\frac{\partial V}{\partial K} + rV}{K^2\frac{\partial^2 V}{\partial K^2}}}$$

where $V(K,t)$ stands for the price of a European call option with strike K and maturity t. Derman and Kani (1996) present local volatility surface as some kind of average forward volatility and relate intuitively the Black-Scholes implied volatility to the volatility surface by means of the relation

$$\sigma^{im}(K,t) = \frac{\int_x^K \tilde{\sigma}(y,t)dy}{K-x}.$$

Once the volatility surface is calibrated to plain vanilla options, Monte Carlo simulations or numerical methods are needed to evaluate more exotic derivatives. In this paper, we present sharp bounds for the price of European options with arbitrary payoff h as well as for barrier and lookback options under the market local volatility surface $\tilde{\sigma}^*(K,t)$. We propose bounds based on the representation of the price as a conditional expectation of a multiplicative functional of a Brownian motion (BM) or a three-dimensional Bessel process (BES) and the simple Jensen's inequality. In the recent literature, the approximation of option price by sharp bounds is widespread. Without claiming any exhaustiveness, we can cite Lo, Yuen and Hui (2003) for barrier options, Vyncke, Goovaerts and Dhaene (2003) for Asian options and Deelstra, Liinev and Vanmaele (2004) for basket options. The application of the Jensen's inequality to the pricing of Asian options was proposed by Rogers and Shi (1995).

In section 2, we present a lower and an upper bound for the conditional expectation of multiplicative functionals of diffusions. In section 3, we describe in a general setting how the bounds can be used to approximate the price of European-style securities. We illustrate numerically the method to price European call option under the CEV model. In sections 4 and 5, we extend our method to the case of double barrier and lookback options for arbitrary local volatility. As by-product result, we obtain a static hedge for lookback option by means a continuum of barrier options. The proofs are brought together in the appendix.

2 Bounds for conditional expectations

Consider the continuous-time diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion, μ and σ are two regular functions. We denote by $\mathbb{Q}_x$ the law of X starting at x and by $E_x^{\mathbb{Q}}$ the expectation relative to $\mathbb{Q}_x$. The diffusion X (pinned) conditioned on $X_t = y$, $t > 0$, has marginal distribution

$$\mathbb{Q}_x(X_s \in dz | X_t = y) = \frac{p(s-t, z, y)p(s, x, z)}{p(t, x, y)} dz \tag{4}$$

where $p(t, x, y)$ is the transition density of the process X. The notation $\mathbb{Q}_{x,y}$ stands for the law of X starting at x and conditioned to arrive in y at time t and $E_{x,y}^{\mathbb{Q}}$ stands for the expectation relative to $\mathbb{Q}_{x,y}$. In this paper, we reduce the pricing of various derivative securities to the computation of the conditional expectation of the exponential process

$$\begin{aligned} Z(t) &= \exp\left\{-\int_0^t V(X_s)ds\right\} \\ &= 1 - \int_0^t V(X_s)e^{-\int_0^s V(X_\tau)d\tau} ds \\ &= 1 - \int_0^t V(X_s)Z(s)ds, \end{aligned} \tag{5}$$

or equivalently, in the differential notation $dZ(t) = -V(X_t)Z(t)dt$. A standard application of Jensen's inequality yields a lower bound for $E_x^{\mathbb{Q}}[Z(t)|X_t = y] = E_{x,y}^{\mathbb{Q}}[Z(t)]$. In the following lemma, we improve the lower bound by an upper bound also based on the simple Jensen's inequality.

Lemma 1 *Let $Z(t)$ be the stochastic process defined by equation (5) where X is a diffusion process taking values in the interval I_X. Assume that it exists a constant α such that $V + \alpha \geq 0$ on I_X. Then, the following inequalities hold*

$$E^-[Z(t)] \leq E_{x,y}^{\mathbb{Q}}[Z(t)] \leq E^+[Z(t)]$$

where the bounds E^- and E^+ are defined by

$$\begin{aligned} E^-[Z(s)] &= \exp\left\{-\int_0^s E_x^{\mathbb{Q}}[V(X_\tau)|X_s]d\tau\right\} \\ E^+[Z(t)] &= e^{-\alpha t} - \int_0^t e^{-\alpha(t-s)} E_{x,y}^{\mathbb{Q}}\left[(V(X_s)+\alpha)E^-[Z(s)]\right] ds. \end{aligned}$$

3 European derivatives

3.1 General framework

A security (or contingent claim) is a financial contract initiated at time 0 that guarantees the holder a future payment whose value is contingent on another underlying asset such as a stock, a bond or even a non-tradable asset as the short-term interest rate or the volatility. If the underlying asset follows a stochastic process $\{X_t, t \geq 0\}$, a European-style contingent claim is defined by a terminal payment $h(X_t)$ at maturity date t and a continuous cash flow $c(X_s, s)$ during the life of the contract ($0 \leq s \leq t$). The price of these contingent claims is the expectation under some risk-neutral measure $\mathbb{Q}$ of the discounted payments, *e.g.* when $c(x, s) = 0$

$$V(x,t) = E_x^{\mathbb{Q}}\left[e^{-\int_0^t r(X_s)ds} h(X_t)\right] \tag{6}$$

where $r(X_s)$ is the instantaneous discount rate. For instance, in the Black-Scholes setting, the discount rate is the risk free interest rate r and the process X is a risky asset such that $dX_t/X_t = rdt + \sigma dW_t$ under the risk-neutral dynamic.

More generally, if the process X is solution of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion, the price $V(x,t)$ of a contingent claim with payout $c(x,s)$ and payoff $h(x)$ can be determined according to the Feynman-Kac theorem as the solution of the following partial differential equation

$$\frac{\partial V}{\partial t} = \mu(x)\frac{\partial V}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 V}{\partial x^2} - r(x)V + c(x,t) \tag{7}$$

with initial condition $V(x,0) = h(x)$. Beaglehole and Tenney (1991) and Jamshidian (1991) solve the general contingent claim pricing equation (7) by means of a kernel function called state-price density. The state-price density denoted by $\Gamma(t,x,y)$ is the particular solution of the pricing equation when $c(x,s)=0$ and $h(x)=\delta(x-y)$ and can be interpreted as the prices of fundamental securities (or Arrow-Debreu securities) that yield 1 only if $X_t=y$ at time to maturity. We can replicate any European contingent claim by purchasing a portfolio of these basic securities and determine its price as

$$V(x,t)=\int_0^t ds \int dy\ \Gamma(s,x,y)c(y,s)+\int dy\ \Gamma(t,x,y)h(y), \tag{8}$$

see e.g. Beaglehole and Tenney (1991). For an arbitrary diffusion process X, the state-price density is in general unknown in closed form. In the next section, we derive a stochastic representation for $\Gamma(t,x,y)$ involving a conditional expectation of a process in the form (5) such that we can apply directly lemma 1 to obtain bounds for the price of any European contingent claims.

3.2 Stochastic representation

We make the same assumptions on the functions μ and σ as Ait-Sahalia (2002) that ensure the existence of the following representation and we consider the two cases relevant in finance where $I_X=(-\infty,+\infty)$ and $I_X=(0,+\infty)$. To start with, we transform X into a unit volatility process $Y=\phi(X)$ where $\phi(x)=\int^x \frac{dz}{\sigma(z)}$. The Itô's lemma yields the stochastic differential equation

$$dY_t=\mu_Y(Y_t)dt+dW_t, \quad Y_0=\phi(x)$$

where

$$\mu_Y(y)=\frac{\mu(\phi^{-1}(y))}{\sigma(\phi^{-1}(y))}-\frac{1}{2}\frac{\partial\sigma}{\partial x}(\phi^{-1}(y)). \tag{9}$$

This transformation is very convenient as Y is more similar to a Gaussian process and I_Y the domain of Y remains either the whole real line or the half line, see Ait-Sahalia (1999).

Lemma 2 *Let $\{W_t, t\geq 0\}$ be a standard $\mathbb{Q}$-Brownian motion and the underlying asset X the solution of the following stochastic differential equation*

$$dX_t=\mu(X_t)dt+\sigma(X_t)dW_t, \quad X_0=x.$$

Then, the state-price density $\Gamma(t,x,y)$ for claims contingent on X can be written as

$$\begin{aligned}\Gamma(t,x,y) &= e^{\int_{\phi(x)}^{\phi(y)} \mu_Y(z)dz} p(t,\phi(x),\phi(y)) \\ &\quad \times E_{\phi(x)} \left[e^{-\int_0^t r\left(\phi^{-1}(B_s)\right)+\lambda_Y(B_s)ds} | B_t = \phi(y) \right] / \sigma(y) \qquad (10)\end{aligned}$$

where $\lambda_Y(y) = \left(\mu_Y^2(y) + \partial\mu_Y(y)/\partial y\right)/2$, $\{B_t, t \geq 0\}$ is a BM in case $I_Y = (-\infty,+\infty)$ or a BES in case $I_Y = (0,+\infty)$ and $p(t;x,y)$ is the transition density of a BM if $I_Y = (-\infty,+\infty)$ or a BM killed at zero if $I_Y = (0,+\infty)$.

Remark 1 *If the instantaneous discount rate $r(.)$ is identically equal to zero, $\Gamma(t,x,y)$ is $p_X(t,x,y)$ the transition probability of the process X.*

3.3 Numerical illustrations

A simple application of lemma 1 and 2 can be found in the pricing of European option on a stock price X. Suppose that the price process $\{X_t, t \geq 0\}$ under the risk-neutral dynamic is solution of the stochastic differential equation

$$\frac{dX_t}{X_t} = (r-d)dt + \tilde{\sigma}(X_t;\theta)dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion, r is the risk free interest rate and d is the dividend rate. The arbitrage free price of a European call option with maturity date t is equal to

$$\begin{aligned}V(x,t) &= e^{-rt} E_x^{\mathbb{Q}}\left[(X_t - K)_+\right] \\ &= \int_0^{+\infty} \Gamma(t,x,y)(y-K)_+ dy \\ &= e^{-rt} \int_K^{+\infty} (y-K)\, p_X(t,x,y)dy \qquad (11)\end{aligned}$$

where $p_X(t,x,y)$ is the transition probability of the process X. For arbitrary local volatility function $\tilde{\sigma}(x)$, no expression for the transition probability of X is available. Cox and Ross (1976) propose the local volatility in the form $\tilde{\sigma}(x) = \sigma x^{\beta}$[1] with $-1 < \beta < 0$. These diffusions are called constant elastic-

[1]For sake of simplicity, we assume that the local volatility is time-homogeneous. However, the results hold for more general seperable local volatility surface $\tilde{\sigma}(x,t) = \varphi(x)\gamma(t)$. This can be shown by a deterministic time change choosing $\exp(rt)$ as numraire.

ity variance [2] models (CEV). For $-1 < \beta < 0$, the origin is an attainable point where we impose an absorbing boundary, the state of absorption corresponding to bankruptcy is attained faster for high value of the dividend rate d. Cox and Ross (1976) obtain an explicit CEV option pricing formula. However, computing the CEV option price formula is not a trivial task as the solution involves an infinite series expansion and the numerical evaluation of an improper integral. Several approximations are proposed in the literature to overcome this drawback, we cite *e.g.* Schroder (1989), Ait-Sahalia (1999) and Lo, Yuen and Hui (2000). In this paper, we propose the bounds for $V(x,t)$ derived from lemma 1 and lemma 2 with $r(.) = 0$ (see Remark 1). We obtain the following inequalities

$$\int_K^{+\infty} (y-K)p_X^-(t,x,y)dy \leq V(x,t)e^{rt} \leq \int_K^{+\infty} (y-K)p_X^+(t,x,y)dy$$

where p_X^- and p_X^+ are respectively the lower and the upper bound for the transition probability of X. Note that the put option (wity payoff $(K-x)_+$) price decomposes into a sum of two terms

$$\begin{aligned} V(x,t)e^{rt} &= A(x,t) + \tilde{V}(x,t)e^{rt} \\ &= K\mathbb{Q}_x(X_t = 0) + \int_{0+}^{K} (K-y)p_X(t,x,y)dy \end{aligned} \tag{12}$$

where $\mathbb{Q}_x(X_t = 0)$ is the probability of absorption bounded as follows

$$1 - \int_{0+}^{+\infty} p_X^+(t,x,y)dy \leq \mathbb{Q}_x(X_t = 0) \leq 1 - \int_{0+}^{+\infty} p_X^-(t,x,y)dy.$$

Figure 1 presents the bounds for the transition probability p_X of the CEV diffusion with parameters $\sigma = 0.2$, $r = 0.06$, $d = 0.02$ and $\beta = -0.25$. The total mass of p_X^- is almost equal to one. This reveals that the probability of absorption is negligible. The bounds for call option prices with spot price $X_0 = 100$ and strike price $K = 120$ is plotted as a function of the maturity date. The (relative) length of the interval defined as

$$\left(V^+(x,t) - V^-(x,t)\right)/V^+(x,t) \tag{13}$$

demonstrates that the error is less than 1% for the largest maturity ($t = 12$ months).

[2] Indeed, the elasticity defined as $\frac{x}{\sigma^2(x)}\frac{d\sigma^2(x)}{dx} = 2\beta + 1$, where $\sigma(x) = x\tilde{\sigma}(x)$ is the instantaneous volatility of the process X, is constant

t (years)	MC	s.e	LB	UB	r.l.
$t = 0.25$	0.0028	$6.99e^{-4}$	0.0029	0.0029	0.0111
$t = 0.5$	0.0817	0.0057	0.0807	0.0818	0.0133
$t = 0.75$	0.3034	0.0122	0.3022	0.3072	0.0162
$t = 1$	0.6551	0.0189	0.6455	0.6583	0.0194

Table 1: Price obtained by Monte Carlo simulation (MC) and the standard error (s.e.) with 10000 paths, the lower bound (LB), the upper bound (UB) and the relative length interval (r.l.)

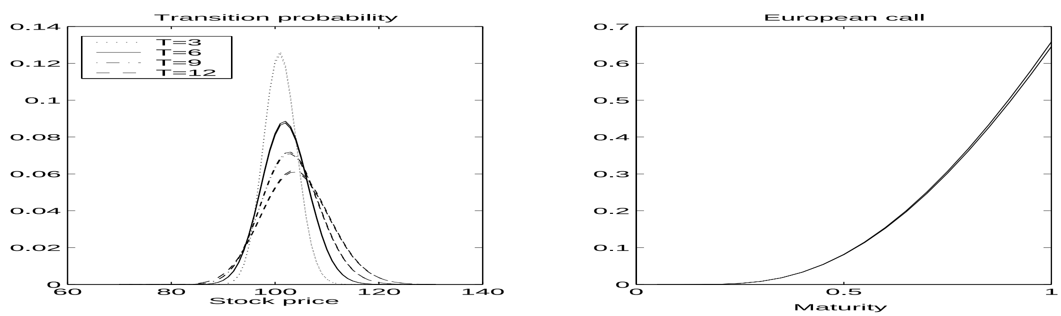

Figure 1: Bounds for the transition probability for different times to maturity (in months) and for the prices of European call option as a function of the maturity (in months) under the CEV model with $X_0 = 100$, $K = 110$, $\sigma = 0.2$, $r = 0.06$, $d = 0.02$ and $\beta = -0.25$

4 Double barrier options

In this section, lemma 1 is applied to the pricing of double barrier options. The holder of a double knock-out barrier option receives a payoff only if the price of the underlying asset remains in an allowable range (L, U). If $\tau_{[L,U]} = \inf\{t \geq 0 : X_t = L \text{ or } X_t = U\}$ is the first hitting time of the stock price process $\{X_t, t \geq 0\}$ at the barriers, the payoff of a double knock-out barrier option with maturity date t is defined as

$$h(X_t) = \Phi(X_t) 1_{(\tau_{[L,U]} > t)} \tag{14}$$

where $\Phi(x) = (x - K)_+$ for a call option or $\Phi(x) = (K - x)_+$ for a put option. Suppose that the price process $\{X_t, t \geq 0\}$ under the risk-neutral dynamic is solution of the stochastic differential equation

$$\frac{dX_t}{X_t} = rdt + \tilde{\sigma}(X_t; \theta) dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion, r is the risk free interest rate and $\tilde{\sigma}(X_t;\theta)$ is the local volatility. The tansformed process $\{Y_t = \phi(X_t), t \geq 0\}$ where $\phi(x) = \int^x \frac{dz}{\sigma(z;\theta)}$ and $\sigma(x;\theta) = x\tilde{\sigma}(x;\theta)$, is solution of the stochastic differential equation

$$dY_t = \mu_Y(Y_t)dt + dW_t, \quad Y_0 = \phi(x)$$

where

$$\mu_Y(y) = \frac{r}{\tilde{\sigma}(\phi^{-1}(y);\theta)} - \frac{1}{2}\frac{\partial \sigma}{\partial x}(\phi^{-1}(y);\theta). \tag{15}$$

The price of a double knock-out option is given in the following lemma.

Lemma 3 *The price of a double knock-out barrier option with payoff $h(X_t)$ at maturity date t is*

$$V(t,x) = \int_{(L,U)} \Gamma(t,x,y)\Phi(y)dy$$

where $X_0 = x$. The following stochastic representation holds for $\Gamma(t,x,y)$

$$\begin{aligned}\Gamma(t,x,y) &= e^{\int_{\phi(x)}^{\phi(y)} \mu_Y(z)dz} p(t,\phi(x),\phi(y)) \\ &\quad \times E_{\phi(x)}\left[e^{-\int_0^t r+\lambda_Y(B_s)\ ds}|B_t = \phi(y)\right]/\sigma(y) \end{aligned} \tag{16}$$

with $\lambda_Y(y) = \left(\mu_Y^2(y) + \partial\mu_Y(y)/\partial y\right)/2$ and $\{B_t, t \geq 0\}$ is a BM on $(\phi(L),\phi(U))$ killed at $\tau_{[L,U]}$ with transition density

$$\begin{aligned} p(t,x,y) &= \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{+\infty} exp\left(-\frac{(x-y-2n(\phi(U)-\phi(L))^2}{2t}\right) \\ &\quad -\frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{+\infty} exp\left(-\frac{(x+y-2L+2n(\phi(U)-\phi(L))^2}{2t}\right). \end{aligned}$$

We apply the bounds of lemma 1 to the conditional expectation in the expression of $\Gamma(t,x,y)$ under the CEV model ($\tilde{\sigma}(x) = \sigma x^\beta$) with $r = 0.04$, $\sigma = 0.2$, $\beta = -0.25$, $L = 90$ and $U = 110$. The valuation of barrier option under the CEV model is not a trivial task, we refer to Davydov and Linetsky (2001). Figure 5 plots the transition probability of the stock price in the range (L,U) together with the probability of hitting the barriers before the maturity of the contract. Figure 6 demonstrates the high accuracy of the bounds for pricing double barrier call options.

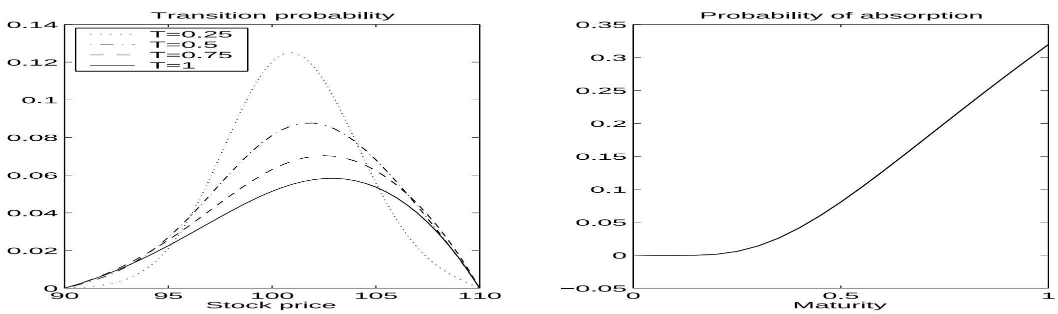

Figure 2: Bounds for the transition probabilities for different maturity dates t, $L = 90$, $U = 110$, $r = 0.04$, $\sigma = 0.2$, $\beta = -0.25$ and $X_0 = 100$.

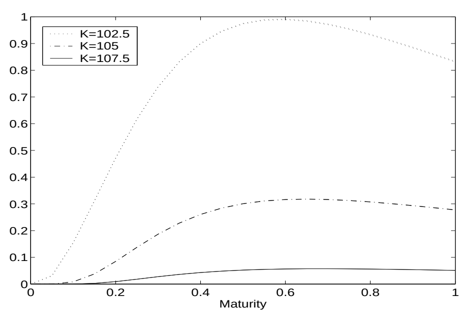

Figure 3: Bounds for the price of a call option for several strike prices K.

5 Lookback options

Lookback options are financial contracts whose payoff involves the running maximum of the underlying stock price defined as

$$M_t = \max_{0 \leq s \leq t} X_s. \tag{17}$$

There is close connection between the running maximum M_t and the first hitting times of the stock process X. As mentioned in Gerber and Shiu (2003), the distribution of the running maximum reads

$$F_{M_t}(a) = \mathbb{Q}_x\left[M_t \leq a\right] = 1 - \mathbb{Q}_x\left[\tau_a \leq t\right]$$

	t (years)	MC	s.e	LB	UB	r.l.
K=102.5	$t = 0.25$	0.6148	0.0127	0.6164	0.6171	0.0011
	$t = 0.5$	0.9748	0.0169	0.9734	0.9753	0.0019
	$t = 0.75$	0.9649	0.0168	0.9530	0.9555	0.0026
	$t = 1$	0.8437	0.0164	0.8309	0.8336	0.0032
K=105	$t = 0.25$	0.1342	0.0055	0.1372	0.1373	$7.28e^{-4}$
	$t = 0.5$	0.3078	0.0085	0.3003	0.3010	0.0023
	$t = 0.75$	0.3192	0.0084	0.3121	0.3131	0.0032
	$t = 1$	0.2853	0.0083	0.2767	0.2777	0.0036
K=107.5	$t = 0.25$	0.0171	0.0014	0.0181	0.0181	0
	$t = 0.5$	0.0464	0.0024	0.0522	0.0524	0.0038
	$t = 0.75$	0.0569	0.0024	0.0568	0.0570	0.0035
	$t = 1$	0.0467	0.0025	0.0509	0.0511	0.0039

Table 2: Price obtained by Monte Carlo simulation (MC) and the standard error (s.e.) with 10000 paths, the lower bound (LB), the upper bound (UB) and the relative length interval (r.l.)

where $\tau_a = \inf\{t \geq 0 : X_t = a\}$. When the price process $\{X_t, t \geq 0\}$ under the risk-neutral dynamic is solution of the stochastic differential equation

$$\frac{dX_t}{X_t} = rdt + \tilde{\sigma}(X_t;\theta)dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a $\mathbb{Q}$-Brownian motion, the price of a lookback call option is given by the following lemma.

Lemma 4 *The price of a lookback call option with maturity date t is given by*

$$V(t,x) = e^{-rt} \int_K^{+\infty} (1 - F_{M_t}(a))\, da$$

where $X_0 = x$. The following stochastic representation holds for $1 - F_{M_t}(a)$

$$\begin{aligned} 1 - F_{M_t}(a) &= \int_{-\infty}^{a} dy\ e^{\int_{\phi(x)}^{\phi(y)} \mu_Y(z)dz} p(t, \phi(x), \phi(y)) \\ &\quad \times E_{\phi(x)} \left[e^{-\int_0^t \lambda_Y(B_s)ds} | B_t = \phi(y) \right] / \sigma(y) \qquad (18) \end{aligned}$$

with $\lambda_Y(y) = \left(\mu_Y^2(y) + \partial\mu_Y(y)/\partial y\right)/2$ and $\{B_t, t \geq 0\}$ is a BM on $(-\infty, \phi(a))$ killed at τ_a with transition density

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} exp\left(-\frac{(y-x)^2}{2t}\right) - \frac{1}{\sqrt{2\pi t}} exp\left(-\frac{(y+x-2\phi(a))^2}{2t}\right).$$

Remark 2 *We can rewrite the price $V(x,t)$ of a lookback option as an integral over a continuum of barrier options:*

$$V(x,t) = \int_K^{+\infty} V_b(x,t)db$$

where $V_b(x,t)$ stands for the price of a single knock-out put option with barrier located at b and strike price b as well. This formula provides a static hedge for lookback option by means of a continuum of single barrier options. The replicating porfolio of barrier options is static in the sense that it ought not to be continuously rebalanced.

Once again, we apply the bounds of lemma 1 to approximate the expectation in expression (18) in case of a CEV diffusion X ($\tilde{\sigma}(x) = \sigma x^\beta$) with $r = 0.04$, $\sigma = 0.2$, $\beta = -0.25$. Figure 7 presents the bounds for the distribution of the running maximum M_t and figure 8 the prices of lookback options for different strikes. Table 5 compares the prices obtained by Monte Carlo simulation with the bounds and reveals that the method is somewhat less efficient for lookback options.

	t (years)	MC	s.e	LB	UB	r.l.
K=102.5	$t = 0.25$	1.0974	0.0163	1.1101	1.1210	0.0097
	$t = 0.5$	2.4519	0.0278	2.4765	2.4980	0.0086
K=105	$t = 0.25$	0.2829	0.0084	0.2841	0.2934	0.0317
	$t = 0.5$	1.1139	0.0202	1.3231	1.3442	0.0157
K=107.5	$t = 0.25$	0.0478	0.0032	0.0357	0.0455	0.2154
	$t = 0.5$	0.4233	0.0129	0.3879	0.4070	0.0469

Table 3: Price obtained by Monte Carlo simulation (MC) and the standard error (s.e.) with 10000 paths, the lower bound (LB), the upper bound (UB) and the relative length interval (r.l.)

Acknowledgements

Marc Decamps and Marc Goovaerts wish to thank the "Onderzoeksraad K.U.Leuven" for a GOA grant.

A Proofs

A.1 Proof of Lemma 1

The lower bound $E^{-}[Z(t)]$ results from a trivial application of Jensen's inequality. Consider the process $\tilde{Z}(t) = e^{-\alpha t}Z(t)$. An application of Itô's lemma yields $d\tilde{Z}(t) = -(V(X_t)+\alpha)\tilde{Z}(t)dt$ or equivalently

$$\tilde{Z}(t) = 1 - \int_0^t (V(X_s)+\alpha)\tilde{Z}(s)ds.$$

Conditioning on the variable X_s, $0 < s < t$, we obtain that

$$\begin{aligned}
E^{\mathbb{Q}}_{x,y}[\tilde{Z}(t)] &= E_{X_s}\left[E^{\mathbb{Q}}_{x,y}[\tilde{Z}(t)|X_s]\right] \\
&= 1 - \int_t^T E_{X_s}\left[(V(X_s)+\alpha)E^{\mathbb{Q}}_{x}[\tilde{Z}(s)|X_s]\right] ds \\
&\leq 1 - \int_t^T E^{\mathbb{Q}}_{x,y}\left[(V(X_s)+\alpha)E^{-}[\tilde{Z}(s)]\right] ds.
\end{aligned}$$

Inserting $E^{-}[\tilde{Z}(s)] = e^{\alpha s}E^{-}[Z(s)]$ and $E[\tilde{Z}(t)] = e^{\alpha t}E^{\mathbb{Q}}_{x,y}[Z(t)]$ provides the upper bound.

A.2 Proof of Lemma 2

We first assume that $I_Y = (-\infty, +\infty)$ and consider the family of contingent claims with payoff function $h(X_t) = 1(X_t \in A)$ for every set A in the σ-field generated by X_t and where $1_{(.)}$ is the indicator function. From martingale arguments as in Artzner and Delbaen (1989), we obtain the arbitrage free prices $V_A(x,t)$ of these securities

$$\begin{aligned}
V_A(x,t) &= E^{\mathbb{Q}}_{x}\left[e^{-\int_0^t r(X_s)ds}\ 1_{(X_t\in A)}\right] \\
&= E^{\mathbb{Q}}_{\phi(x)}\left[e^{-\int_0^t r\left(\phi^{-1}(Y_s)\right)ds}\ 1_{(\phi^{-1}(Y_t)\in A)}\right]
\end{aligned}$$

where Y is the transformed process and $\mathbb{Q}$ is the risk neutral measure. An application of the Girsanov's theorem and the Itô's formula yields

$$V_A(x,t) = E_{\phi(x)}\left[e^{-\int_0^t r\left(\phi^{-1}(B_s)\right)ds-\int_0^t \frac{1}{2}\mu_Y^2(B_s)ds+\int_0^t \mu_Y(B_s)dB_s}\ 1_{(\phi^{-1}(B_t)\in A)}\right]$$

$$
\begin{aligned}
&= E_{\phi(x)}\left[e^{\int_{Y_0}^{Y_t}\mu_Y(z)dz}e^{-\int_0^t r\left(\phi^{-1}(B_s)\right)ds-\int_0^t \lambda_Y(B_s)ds}\ 1_{(\phi^{-1}(B_t)\in A)}\right] \\
&= E_{\phi(x)}\left[e^{\int_{Y_0}^{Y_t}\mu_Y(z)dz}D(t)\ 1_{(\phi^{-1}(B_t)\in A)}\right].
\end{aligned}
$$

If we define $\Phi(y) = E_{\phi(x)}\left[D(t)|Y_t = \phi(y)\right]$, $V_A(x,t)$ simplifies as

$$
V_A(x,t) = \int_A e^{\int_{\phi(x)}^{\phi(y)}\mu_Y(z)dz}\ \Phi(y)\ \frac{1}{\sqrt{2\pi t}}e^{-\frac{(\phi(y)-\phi(x))^2}{2t}}d\phi(y).
$$

The result follows by identification with the integral formula (8) of the derivative prices for any set A.

We briefly discuss the case where $I_Y = (0,+\infty)$. We can decompose the drift of Y as the sum $\mu_Y(x) = \left(\mu_Y(y) - \frac{1}{y}\right) + \frac{1}{y}$ and apply the Girsanov's theorem to reduce Y to a BES. The law of the BM killed at 0 arises as

$$
p(t,x,y) = \frac{y}{x}g(t,x,y)
$$

where $g(t,x,y)$ is the transition density of the BES.

A.3 Proof of Lemma 3

The holder of a double barrier option receives a payoff only if the underlying asset remains in the range (L,U). The price of a barrier option is the expectation of the payoff under the risk-neutral measure conditional on the event $\{\tau_{[L,U]} < t\}$ where $\tau_{[L,U]}$ is the first hitting time at the barriers, thus

$$
\begin{aligned}
V(x,t) &= e^{-rt}E_x^{\mathbb{Q}}\left[h(X_t)|\tau_{[L,U]} < t\right] \\
&= e^{-rt}E_x^{\mathbb{Q}}\left[h\left(\phi^{-1}(Y_t)\right)|\tau_{[L,U]} < t\right].
\end{aligned}
$$

However, we can still apply the Girsanov's theorem, but the Radon-Nikodym derivative

$$
D(t) = e^{-\int_0^t \frac{1}{2}\mu_Y^2(B_s)ds+\int_0^t \mu_Y(B_s)dB_s}
$$

with $t < \tau_{[L,U]}$ is only a local martingale. The process Y is then reduced to a Brownian motion killed at $\tau_{[L,U]}$. Conditioning on $Y_t = \phi(y)$ provides the result with the same calculation as in the proof of lemma 2 as $\tau_{[L,U]} = \inf\{t \geq 0 : Y_t = \phi(L)\ or\ Y_t = \phi(U)\}$.

A.4 Proof of Lemma 4

By partial integration, we can prove that

$$\begin{aligned} V(x,t) &= e^{-rt} E_x^{\mathbb{Q}} \left[(M_t - K)_+ \right] \\ &= e^{-rt} \int_K^{+\infty} p_M(t,x,y)(y-K) dy \\ &= e^{-rt} \int_K^{+\infty} (1 - F_{M_t}(a)) da. \end{aligned}$$

$F_{M_t}(a)$ is the distribution function of the running maximum $M_t = \max_{0 \leq s \leq t} X_s$ and can be expressed as

$$\begin{aligned} 1 - F_{M_t}(a) &= Pr \left[\tau_a \leq t \right] \\ &= \int_{-\infty}^{a} p_X(t,x,y) dy \end{aligned}$$

where $p_X(t,x,y)$ is the transition density of the process X absorbed at a. An application of lemma 2 to $p_X(t,x,y)$ provides the desired result.

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