Unified Pricing of Asian Options

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Abstract. A simple and numerically stable 2-term partial differential equation characterizing the price of any type of arithmetically averaged Asian option is given. The approach includes both continuously and discretely sampled options and it is easily extended to handle continuous or discrete dividend yields. In contrast to present methods, this approach does not require to implement jump conditions for sampling or dividend days.

Asian options are securities with payoff which depends on the average of the underlying stock price over certain time interval. Since no general analytical solution for the price of the Asian option is known, a variety of techniques have been developed to analyze arithmetic average Asian options. There is enormous literature devoted to study of this option. A number of approximations that produce closed form expressions have appeared, most recently in Thompson (1999), who provides tight analytical bounds for the Asian option price. Geman and Yor (1993) computed the Laplace transform of the price of continuously sampled Asian option, but numerical inversion remains problematic for low volatility and/or short maturity cases as shown by Fu, Madan and Wang (1998). Very recently, Linetsky (2002) has derived new integral formula for the price of continuously sampled Asian option, which is again slowly convergent for low volatility cases. Monte Carlo simulation works well, but it can be computationally expensive without the enhancement of variance reduction techniques and one must account for the inherent discretization bias resulting from the approximation of continuous time processes through discrete sampling as shown by Broadie, Glasserman and Kou (1999).

In general, the price of an Asian option can be found by solving a partial differential equation (PDE) in two space dimensions (see Ingersoll (1987)), which is prone to oscillatory solutions. Ingersoll also observed that the two-dimensional PDE for a floating strike Asian option can be reduced to a one-dimensional PDE. Rogers and Shi (1995) have formulated a one-dimensional PDE that can model both floating and fixed strike Asian options. However this one-dimensional PDE is difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid. There are several articles which try to improve the numerical performance of this PDE type. Andreasen (1998) applied Rogers and Shi's reduction to discretely sampled Asian option.

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There are several independent efforts in recent years to unify pricing techniques for different types of options and relate these methods to pricing Asian option. Lipton (1999) noticed similarity of pricing equations for the passport, lookback and the Asian option, again using Rogers and Shi's reduction. Shreve and Vecer (2000) developed techniques for pricing options on a traded account, which include all options which could be replicated by a self-financing trading in the underlying asset. These option include European, passport, vacation, as well as Asian options. Numerical techniques for pricing contracts of this type are described in Vecer (2000). Hoogland and Neumann (2001) developed alternative framework for pricing various types of options using scale invariance methods and derived more general semianalytic solutions for prices of continuously sampled Asian options.

This paper provides even simpler and unifying approach for pricing Asian options, for both discrete and continuous arithmetic average. The resulting one-dimensional PDE for the price of the Asian option can be easily implemented to give extremely fast and accurate results. Moreover, this approach easily incorporates cases of continuous or discrete dividends.

Replication of Asian Forward

Suppose that the underlying asset evolves under the risk neutral measure according to the equation

$$dS_t = S_t((r-\gamma)dt + \sigma dW_t), \tag{1}$$

where r is the interest rate, γ is a continuous dividend yield, and σ is the volatility of the underlying asset. Denote the trading strategy by q_t, the number of shares held at time t. If we take the strategy q_t to be

$$q_t = \tfrac{1}{(r-\gamma)T}(e^{-\gamma(T-t)} - e^{-r(T-t)}) \tag{2}$$

and let the wealth evolve according to the following self-financing strategy

$$\begin{aligned} dX_t &= q_t dS_t + r(X_t - q_t S_t)dt + q_t \gamma S_t dt & (3)\\ &= rX_t dt + q_t(dS_t - rS_t dt + \gamma S_t dt) & (4) \end{aligned}$$

with the initial wealth

$$X_0 = \tfrac{1}{(r-\gamma)T}(e^{-\gamma T} - e^{-rT})S_0, \tag{5}$$

we have

$$\begin{aligned} X_T &= e^{rT}X_0 + \int_0^T q_t e^{r(T-t)}(dS_t - rS_t dt + \gamma S_t dt) & (6)\\ &= e^{rT}X_0 + q_T S_T - e^{rT} q_0 S_0 + \int_0^T e^{r(T-t)} S_t (q_t \gamma dt - q_t' dt) & (7)\\ &= \tfrac{1}{T}\int_0^T S_t dt, & (8) \end{aligned}$$

because of the identity

$$d(e^{r(T-t)}q_t S_t) = e^{r(T-t)}q_t dS_t - re^{r(T-t)}q_t S_t dt + e^{r(T-t)}S_t dq_t, \tag{9}$$

or

$$q_T S_T - e^{rT}q_0 S_0 = \int_0^T e^{r(T-t)}q_t(dS_t - rS_t dt) + \int_0^T e^{r(T-t)}S_t dq_t. \tag{10}$$

Note that the above analysis does not require specification of the stock dynamics and thus is model independent. Asian forward can be replicated in the exactly same way for stocks exhibiting alternative dynamics, like stochastic volatility or jumps. It is possible to extend the technique presented here for pricing Asian options on stocks driven by jump diffusion models, as shown in Vecer and Xu (2002).

Relationship to Asian options

Using this idea that we can replicate the stock price average by self-financing trading in the stock, we can apply this fact to pricing Asian options. The general payoff of the Asian option could be written as

$$(\bar{S}_T - K_1 S_T - K_2)^+ \quad \text{or} \quad (K_2 - K_1 S_T - \bar{S}_T)^+. \tag{11}$$

Because of the Asian Put-Call parity

$$\begin{aligned} & e^{-rT}\mathbb{E}(\bar{S}_T - K_1 S_T - K_2)^+ - e^{-rT}\mathbb{E}(K_2 - K_1 S_T - \bar{S}_T)^+ \\ &\quad = e^{-rT}\mathbb{E}(\bar{S}_T - K_1 S_T - K_2) = \tfrac{1}{(r-\gamma)T}(e^{-\gamma T} - e^{-rT}) \cdot S_0 - K_1 e^{-\gamma T} S_0 - e^{-rT} K_2, \end{aligned} \tag{12}$$

it is enough to compute the value of the Asian option with the payoff $(\bar{S}_T - K_1 S_T - K_2)^+$. In this case, when $K_1 = 0$, then we have the fixed strike Asian call option, when $K_2 = 0$, then we have the floating strike Asian put option. In order to replicate such option, hold at time t

$$q_t = \tfrac{1}{(r-\gamma)T}(e^{-\gamma(T-t)} - e^{-r(T-t)}) \tag{13}$$

of the stock, start with the initial wealth

$$X_0 = q_0 S_0 - e^{-rT} K_2 \tag{14}$$

and follow the self-financing strategy

$$dX_t = q_t dS_t + r\,(X_t - q_t S_t)\,dt + q_t \gamma S_t dt. \tag{15}$$

It turns out that $X_T = \bar{S}_T - K_2$ and the payoff of the option is then

$$(X_T - K_1 S_T)^+ = (\bar{S}_T - K_1 S_T - K_2)^+. \tag{16}$$

We can use the change of numeraire technique to reduce dimensionality of the problem by defining

$$Z_t = \frac{X_t}{e^{\gamma t} S_t}. \tag{17}$$

According to Ito's lemma,

$$\begin{aligned} dZ_t &= \left(Z_t - e^{-\gamma t} q_t\right)\sigma^2 dt - \left(Z_t - e^{-\gamma t} q_t\right)\sigma dW_t && (18) \\ &= -\left(Z_t - e^{-\gamma t} q_t\right)\sigma d\widetilde{W}_t, && (19) \end{aligned}$$

where $\widetilde{W}_t = -\sigma t + W_t$ is a Brownian motion under the numeraire measure. The price of Asian call option could be written as

$$V(0, S_0, K_1, K_2) = e^{-rT}\mathbb{E}(X_T - K_1 S_T)^+ = S_0 \cdot \widetilde{\mathbb{E}}(Z_T - K_1)^+. \tag{20}$$

If we introduce

$$u(0, Z_0) = \widetilde{\mathbb{E}}(Z_T - K_1)^+, \tag{21}$$

where Z_t is a process defined by (19) with the initial condition

$$Z_0 = \frac{X_0}{S_0} = \frac{1}{(r-\gamma)T}(e^{-\gamma T} - e^{-rT}) - e^{-rT}\frac{K_2}{S_0}, \tag{22}$$

then the price of the option is

$$V(0, S_0, K_1, K_2) = S_0 \cdot u(0, Z_0). \tag{23}$$

It is easy to show that the function u satisfies the following partial differential equation

$$u_t + \frac{1}{2}\left(z - e^{-\gamma t} q_t\right)^2 \sigma^2 u_{zz} = 0, \tag{24}$$

$$u(T, z) = (z - K_1)^+. \tag{25}$$

This unconditionally stable equation could be easily solved numerically by the finite difference method. Similar partial differential equation was previously derived for a more general case of the option on a traded account in Shreve and Vecer (2000) and subsequently in Hoogland and Neumann (2001) in their independent work.

Generally Sampled Options on Stocks with General Dividends

The same technique could be applied for pricing discretely sampled Asian option, or for Asian options with averaging with different weighting factors. We can also incorporate the case of general dividends (continuous or discrete) in the evolution of the stock price; we can assume that the stock price has the following dynamics:

$$dS_t = S_t(rdt - d\nu_t + \sigma dW_t) \tag{26}$$

under the risk neutral measure and ν_t is the measure representing the dividend yield. For example, when $d\nu_t = \gamma dt$, we have a continuous dividend yield at the rate γ.

Suppose that we want to price option whose payoff depends on

$$\int_0^T S_t d\mu(t), \tag{27}$$

where $\mu(t)$ represents a general weighting factor. For continuous averaging we had $d\mu(t) = \frac{1}{T}dt$, for discrete averaging (n points) we can set

$$d\mu(t) = \frac{1}{n}\sum_{k=1}^{n} \delta(\tfrac{k}{n}T)dt, \tag{28}$$

so that $\int_0^T S_t d\mu(t)$ becomes $\frac{1}{n}\sum_{k=1}^{n} S_{\frac{k}{n}T}$ - the discretely sampled average. Here $\delta(\cdot)$ is the Dirac delta function. In order to replicate the payoff depending on (27), we would like to find a trading strategy q_t such that

$$X_T = \int_0^T S_t d\mu(t) - K_2 \tag{29}$$

as in the previous section. The wealth evolves according to

$$dX_t = q_t dS_t + r\left(X_t - q_t S_t\right)dt + q_t S_t d\nu_t, \tag{30}$$

and therefore

$$X_T = e^{rT}X_0 + \int_0^T e^{r(T-t)} q_t (dS_t - rS_t dt + S_t d\nu_t). \tag{31}$$

We can use the identity

$$d\left[e^{r(T-t)}q_tS_t\right] = e^{r(T-t)}q_t dS_t - re^{r(T-t)}q_tS_t dt + e^{r(T-t)}S_t dq_t \tag{32}$$

to simplify the expression for X_T:

$$X_T = e^{rT}X_0 + q_TS_T - e^{rT}q_0S_0 + \int_0^T e^{r(T-t)}S_t(q_t d\nu_t - dq_t). \tag{33}$$

To get the desired result (29), we set $q_T = 0$, $e^{rT}(X_0 - q_0S_0) = -K_2$ and $e^{r(T-t)}(q_t d\nu_t - dq_t) = d\mu(t)$. This leads to the following representation of the trading strategy:

$$q_t = \exp\left(-\int_t^T d\nu(s)\right)\cdot\int_t^T \exp\left(-r(T-s) + \int_s^T d\nu(u)\right)d\mu(s), \tag{34}$$

which is a result consistent with the previous section.

In particular, when the stock does not pay any dividends ($d\nu_t = 0$), we have

$$q_t = e^{-rT}\int_t^T e^{rs}d\mu(s). \tag{35}$$

For the case of discretely sampled options and no dividends we have $d\mu(t) = \frac{1}{n}\sum_{k=1}^n \delta(\frac{k}{n}T)dt$, and therefore

$$q_t = \tfrac{1}{n}e^{-rT}\int_t^T e^{rs}\sum_{k=1}^n \delta(\tfrac{k}{n}T)ds = \tfrac{1}{n}\sum_{k=\left[\frac{nt}{T}\right]+1}^{n} \exp\left(-r(\tfrac{n-k}{n})T\right), \tag{36}$$

where $[\cdot]$ denotes the integer part function. The strategy which replicates discretely sampled average is just a step function with jumps during sampling dates, and it is a discrete approximation of the strategy for continuous average.

We conclude that the price of the Asian option with the payoff $(\int_0^T S_t d\mu(t) - K_1S_T - K_2)^+$ on a stock paying arbitrary dividends at the rate $d\nu_t$ is given by

$$V(0, S_0, K_1, K_2) = S_0 \cdot u(0, Z_0), \tag{37}$$

where $u(t, z)$ satisfies

$$u_t + \tfrac{1}{2}\left(z - e^{-\int_0^t d\nu(s)}q_t\right)^2\sigma^2 u_{zz} = 0, \tag{38}$$

$$u(T, z) = (z - K_1)^+. \tag{39}$$

The strategy q_t is given by (34) and $Z_0 = \frac{X_0}{S_0} = q_0 - e^{-rT}\frac{K_2}{S_0}$. The implementation differs from the continuous average case only by the choice of the strategy.

It is worth noting the fact that the above PDE has a few obvious closed form solutions. For example $u(t, z) = 1$ is a solution with the boundary condition $u(T, z) = 1$, and $u(t, z) = z$ is a solution with the boundary condition $u(T, z) = z$. It is not very difficult to generate more closed form solutions for polynomial boundary conditions. Having enough closed form solutions in hand, one can significantly enhance the numerical procedure for solving Asian option price.

Comparison to Other Methods

Present popular techniques for pricing Asian options include: Monte Carlo simulation, numerical inversion of the Laplace transform of the Asian option price derived by Geman and Yor (1993), alternative PDE techniques suggested by Rogers and Shi (1995), and various approximations (for instance, Turnbull and Wakeman (1992)). Numerical inversion of the Laplace transform or PDE techniques of Rogers and Shi tend to give slowly convergent results for cases of short maturities or small volatilities. These techniques could be improved, as shown for instance by Fu, Madan and Wang (98/99) or by Zvan, Forsyth and Vetzal (97/98), but at the expense of the speed and the complexity of the implementation. Monte Carlo simulation gives accurate results for all choices of parameters. Although the speed of Monte Carlo simulation could be further enhanced by specific choice of control variates, it is inherently computationally inefficient to price Asian options.

Since all present techniques are either computationally unstable, too slow or too complicated to implement, there had not been a single technique which would be widely accepted to price Asian options for all choices of market parameters. In contrast, the method presented in this paper is stable for all choices of parameters (even for small volatilities and short maturities) and it gives accurate results within 6 decimal digits in less than a second of computation time.

Table 1 gives a comparison of numerical results of this method with other techniques for pricing continuously sampled Asian option. These techniques include: Geman–Eydeland (GE), Turnbull–Wakeman (TW) and Monte Carlo (MC) results based on 10 daily readings and 100 daily readings with standard errors based on 10 000 replications given in parentheses. The inversion of the Laplace transform method of Geman–Eydeland was implemented with 6 digit precision by Shaw (2000) and the result coincides with the prices of our PDE technique. These numbers also match the very recent result of Linetsky (2002) who computed these prices with even higher precision using his integral formula. Numerical results of techniques of Turnbull–Wakeman and Monte–Carlo methods were reported in Fu, Madan and Wang (98/99). The comparison shows that the PDE method suggested in this paper is consistent with all other methods.

r	σ	T	$S(0)$	Vecer, GE (Shaw), Linetsky	TW	MC10	MC100	Std Err
0.05	0.5	1	1.9	0.193174	0.195	0.192	0.196	(0.004)
0.05	0.5	1	2.0	0.246416	0.250	0.245	0.249	(0.004)
0.05	0.5	1	2.1	0.306220	0.311	0.305	0.309	(0.005)
0.02	0.1	1	2.0	0.055986	.0568	.0559	.0565	(.0008)
0.18	0.3	1	2.0	0.218388	0.220	0.219	0.220	(0.003)
.0125	0.25	2	2.0	0.172269	0.173	0.173	0.172	(0.003)
0.05	0.5	2	2.0	0.350095	0.359	0.351	0.348	(0.007)

Table 1: Comparison of numerical techniques for pricing continuous Asian call option: Vecer, Geman-Eydeland (GE), Linetsky, Turnbull-Wakeman (TW), Monte Carlo (MC). The fixed strike is $K = 2$, T is in years, no dividends.

The advantage of the method presented here is that it could be also used to price discretely sampled Asian option. Table 2 gives a comparison of this method to results presented in Tavella and Randall (2000) and in Curran (1995). Tavella and Randall use PDE techniques of Rogers and Shi with jump conditions, while Curran gives geometrical conditioning approximation. As in the case of continuously sampled Asian options, we get consistent results in less than a second of computation time.

As another example of flexibility of this method, we consider discretely sampled Asian call option with 125 sampling points on a stock paying discrete dividend. Prices from the PDE method are comparable to results of Monte Carlo simulation as shown in Table 3. It is not surprising that the sooner is the dividend payment, the lower is the Asian option price. The computation based on the PDE method took less than a second of computation time. On the other hand, Monte Carlo simulation implementation with control variate reduction techniques took more than 10 minutes

S(0)	Samples	Vecer	Tavella-Randall	Curran
95	10	9.2228	9.2149	9.2197
	25	8.7080	8.6974	8.7053
	50	8.5367	8.5383	8.5340
	125	8.4339	8.4304	8.4314
	250	8.4001	8.3972	8.3972
	500	8.3826	8.3804	8.3801
	1000	8.3741	8.3719	8.3715
	∞	8.3661	8.3640	–
100	10	12.0420	12.0348	12.0390
	25	11.4906	11.4803	11.4881
	50	11.3068	11.2982	11.3043
	125	11.1967	11.1929	11.1940
	250	11.1600	11.1573	11.1572
	500	11.1416	11.1392	11.1388
	1000	11.1322	11.1300	11.1296
	∞	11.1233	11.1215	–
105	10	15.2234	15.2168	15.2202
	25	14.6510	14.6415	14.6483
	50	14.4601	14.4519	14.4575
	125	14.3455	14.3424	14.3430
	250	14.3073	14.3054	14.3048
	500	14.2881	14.2866	14.2857
	1000	14.2786	14.2771	14.2762
	∞	14.2696	14.2681	–

Table 2: Comparison of numerical techniques for discretely sampled Asian call option, $r = 0.1$, $\sigma = 0.4$, $T = 1$, fixed strike $K = 100$, no dividends.

to get the prices within 0.01 standard error.

Pricing Asian Options after Starting Date

We can easily modify this approach to price the situation where the option is not newly issued and some prices used to determine the average have already been observed. Suppose that the contract was initiated at time 0, expires at time T with the payoff $(\bar{S}_T - K_1 S_T - K_2)^+$, and we observe it at time t. The price at time t is then

$$\begin{aligned} &e^{-r(T-t)}\mathbb{E}_t(\bar{S}_T - K_1 S_T - K_2)^+ = e^{-r(T-t)}\mathbb{E}_t(\tfrac{1}{T}\int_t^T S_s ds + \tfrac{t}{T}\bar{S}_t - K_1 S_T - K_2)^+ = \\ &e^{-r(T-t)}\tfrac{T-t}{T}\mathbb{E}_t(\bar{S}_{T-t} - \tfrac{K_1 T}{T-t} S_T - (\tfrac{T}{T-t}K_2 - \tfrac{t}{T-t}\bar{S}_t))^+ = \tfrac{T-t}{T} \cdot V(t, S_t, \tfrac{K_1 T}{T-t}, \tfrac{T}{T-t}K_2 - \tfrac{t}{T-t}\bar{S}_t). \end{aligned} \tag{40}$$

This is a value of an Asian call option with different strikes $\widehat{K}_1 = \frac{K_1 T}{T-t}$, $\widehat{K}_2 = \frac{T}{T-t}K_2 - \frac{t}{T-t}\bar{S}_t$.

American Asians

It should be noted that this technique does not enable to solve for the American Asian style option. The reason is that the pricing equation (24) explicitly depend on the maturity of the

$S(0)$	dividend day	Vecer	Monte Carlo
105	0.25	9.5688	9.5609
	0.50	10.9470	10.9618
	0.75	12.5335	12.5455
100	0.25	7.1974	7.2057
	0.50	8.3238	8.3323
	0.75	9.6522	9.6418
95	0.25	5.2026	5.1978
	0.50	6.0879	6.1026
	0.75	7.1605	7.1788

Table 3: Comparison of our technique to Monte Carlo simulation for discretely sampled Asian option with 125 sampling points on a stock paying discrete dividend at the rate 10%, $T = 1$, $r = 0.1$, $\sigma = 0.4$, fixed strike $K = 100$. Standard errors for Monte Carlo prices are in all cases equal to 0.01.

option and the price of the American style option cannot be related to the pricing function at any other time than at expiration. Consequently, in order to solve the American style problem, we would have to keep track of both the stock price and its running average, which leads back to the two-dimensional formulation of the problem. However, as pointed out by Andreasen (1998), it is possible to reduce the dimensionality of the American Asian option pricing equation for the case of the floating strike by introducing a new reduction space variable $\frac{\bar{S}_t}{S_t}$, the ratio of the running average and the stock price. The case of Amerian Asian option cannot be reduced to one dimension for the case of the fixed strike.

Conclusion

The pricing method for Asian options suggested in this article unifies pricing of arithmetic average Asian options of both discrete and continuous types. The method suggested here has a simple form, is easy to implement, has stable performance for all volatilities, and is fast and accurate. It is also shown how to price easily options on dividend paying stocks.

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